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Approximation formulas for the constant $e$ and an improvement to a Carleman-type inequality

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Abstract. We give an explicit formula for the determination of the coefficients $c_j$ appearing in the expansion

$$x \left( 1 + \sum_{j=1}^{q} \frac{c_j}{x^j} \right) \left( \frac{\sqrt{\pi}}{\Gamma(x + \frac{1}{2})} \right)^{1/x} = e + O \left( \frac{1}{x^{q+1}} \right)$$

for $x \to \infty$ and $q \in \mathbb{N} := \{1, 2, \ldots\}$. We also derive a pair of recurrence relations for the determination of the constants $\lambda_{\ell}$ and $\mu_{\ell}$ in the expansion

$$(1 + \frac{1}{x})^x \sim e \left( 1 + \sum_{\ell=1}^{\infty} \frac{\lambda_{\ell}}{(x + \mu_{\ell})^{2\ell - 1}} \right)$$

as $x \to \infty$. Based on this expansion, we establish an inequality for $(1 + 1/x)^x$. As an application, we give an improvement to a Carleman-type inequality.

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1 Introduction

The constant $e$ can be defined by the limit

$$e = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x.$$
With the possible exception of \(\pi\), \(e\) is the most important constant in mathematics since it appears in myriad mathematical contexts involving limits and derivatives. Joost Bürgi seems to have been the first to formulate an approximation to \(e\) around 1620, obtaining three-decimal-place accuracy (see [13, p. 31], [23], and [29, pp. 26–27]).

Theorems 1.1 and 1.2 below were proved by Chen and Mortici [9].

**Theorem 1.1.** For all \(n \in \mathbb{N} := \{1, 2, 3, \ldots\}\),

\[
2 \left( n + \alpha \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n} < e \leq 2 \left( n + \beta \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n}
\]

(1.1)

with the best possible constants

\[
\alpha = \frac{\ln 2}{2} = 0.34657 \ldots \quad \text{and} \quad \beta = \frac{e}{2} - 1 = 0.35914 \ldots
\]

**Theorem 1.2.** Let \((v_n)_{n \in \mathbb{N}}\) be defined by

\[
v_n = 2 \left( n + \frac{\ln 2}{2} + \frac{a n + b}{n^2} \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n}.
\]

(1.2)

Then, for

\[
a = \frac{3(\ln 2)^2 - 1}{24}, \quad b = \frac{(\ln 2)^3 - \ln 2}{48},
\]

we have

\[
\lim_{n \to \infty} n^4 (v_n - e) = -\frac{e(19 - 30(\ln 2)^2 + 15(\ln 2)^4)}{5760}.
\]

The speed of convergence of the sequence \((v_n)_{n \in \mathbb{N}}\) is \(n^{-4}\).

By using the Maple software, we find, as \(n \to \infty\),

\[
2n \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = e + O \left( \frac{1}{n} \right),
\]

(1.3)

\[
2n \left( 1 + \frac{\ln 2}{2n} \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = e + O \left( \frac{1}{n^2} \right),
\]

(1.4)

\[
2n \left( 1 + \frac{\ln 2}{2n} + \frac{3(\ln 2)^2 - 1}{24n^2} \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = e + O \left( \frac{1}{n^3} \right)
\]

(1.5)

and

\[
2n \left( 1 + \frac{\ln 2}{2n} + \frac{3(\ln 2)^2 - 1}{24n^2} + \frac{(\ln 2)^3 - \ln 2}{48n^3} \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = e + O \left( \frac{1}{n^4} \right).
\]

(1.6)

Motivated by (1.3)-(1.6), we first establish a general approximation formula for \(e\) (given in Theorem 2.1, by mainly using the partition function. From this result, we give an explicit formula for the coefficients \(c_j\) \((1 \leq j \leq q)\) such that

\[
2n \left( 1 + \sum_{j=1}^{q} \frac{c_j}{n^j} \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = e + O \left( \frac{1}{n^{q+1}} \right)
\]

(1.7)
for \( n \to \infty \) and \( q \in \mathbb{N} \), which contains the formulas (1.3)-(1.6) as special cases.

The second aim of the paper is to derive a pair of recurrence relations for the determination of the constants \( \lambda_\ell \) and \( \mu_\ell \) in the expansion
\[
\left(1 + \frac{1}{x}\right)^x \sim e \left(1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(x + \mu_\ell)^{2\ell - 1}}\right)
\]
as \( x \to \infty \) (given in Theorem 3.1). Based on this expansion, we establish an inequality for \((1 + 1/x)^x\) and, as an application, we give an improvement to a Carleman-type inequality (Remark 3.2).

2 The general form of the coefficients \( c_j \) in (1.7)

For our later use, we introduce the following set of partitions of an integer \( n \in \mathbb{N} \):
\[
\mathcal{A}_n := \{(k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n : k_1 + 2k_2 + \cdots + nk_n = n\} .
\]
(2.1)

In number theory, the partition function \( p(n) \) represents the number of possible partitions of \( n \in \mathbb{N} \); that is, the number of distinct ways of representing \( n \) as a sum of natural numbers (with order irrelevant). By convention \( p(0) = 1 \) and \( p(n) = 0 \) for \( n \) negative integers. For more information on the partition function \( p(n) \), see [38] and the references therein. The first few values of the partition function \( p(n) \) are (starting with \( p(0) = 1 \)) (see [37]):

\[
1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots
\]

It is easy to see that the cardinality of the set \( \mathcal{A}_n \) is equal to the partition function \( p(n) \).

The following results are needed in our present investigation. The logarithm of the gamma function has the asymptotic expansion (see [28, p. 32]):
\[
\ln \Gamma(x + t) \sim \left(x + t - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^n B_{n+1}(t)}{n(n+1)} \frac{1}{x^n}
\]
as \( x \to \infty \), where \( B_n(t) \) denotes the Bernoulli polynomials defined by the following generating function:
\[
x^t e^x = e^{x+t} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} .
\]
(2.3)

Note that the Bernoulli numbers \( B_n \) are defined by \( B_n := B_n(0) \) in (2.3).

Taking \( t = \frac{1}{2} \) in (2.2), we have
\[
\ln \Gamma\left(x + \frac{1}{2}\right) \sim x \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} (-1)^n B_{n+1}(\frac{1}{2}) \frac{1}{x^n}
\]
as \( x \to \infty \). Noting that
\[
B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1)B_n \quad \text{for} \quad n \in \mathbb{N}_0
\]
(see [1, p. 805, 23.1.21]), we find from (2.4) that
\[
1 + \frac{1}{x} \ln \Gamma\left(x + \frac{1}{2}\right) - \ln x - \frac{1}{2x} \ln(\pi) = \frac{\ln 2}{2x} + \sum_{j=2}^{q} \frac{(-1)^{j-1}(1 - 2^{1-j})B_j}{(j-1)j} \frac{1}{x^j} + O\left(\frac{1}{x^{q+1}}\right)
\]
as \( x \to \infty \).
Theorem 2.1. The following approximation formula for the constant $e$ holds true:

$$x \left(1 + \sum_{j=1}^{q} \frac{c_j}{x^j}\right) \left(\frac{\sqrt{\pi}}{\Gamma\left(x + \frac{1}{2}\right)}\right)^{1/x} = e + O\left(\frac{1}{x^{q+1}}\right)$$  \hspace{1cm} (2.6)

for $x \to \infty$ and $q \in \mathbb{N}$, with the coefficients $c_j$ ($1 \leq j \leq q$) given by

$$c_j = (-1)^j \sum_{(k_1, k_2, \ldots, k_q) \in A_j} \frac{(-1)^{k_1+k_2+\cdots+k_q}}{k_1! k_2! \cdots k_q!} S_1^{k_1} S_2^{k_2} \cdots S_j^{k_j},$$  \hspace{1cm} (2.7)

where the $A_j$ ($for \ j \in \mathbb{N}$) are given in (2.1), $S_1 = \frac{\ln 2}{2}$, $S_j = \frac{(1 - 2^{1-j})B_j}{j-1}$ ($2 \leq j \leq q$), and $B_n$ are the Bernoulli numbers.

Proof. To determine the coefficients $c_j$ ($1 \leq j \leq q$), we first express (2.6) in the form

$$\ln \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_q}{x^q}\right) = \ln \frac{2}{x} + \sum_{j=2}^{q} \frac{(-1)^{j-1}(1 - 2^{1-j})B_j}{(j-1)j} \frac{1}{x^j} + O\left(\frac{1}{x^{q+1}}\right)$$  \hspace{1cm} (2.8)

as $x \to \infty$, upon making use of (2.5). From the fundamental theorem of algebra, we see that there exist unique complex numbers $x_1, \ldots, x_q$ such that

$$1 + \frac{c_1}{x} + \cdots + \frac{c_q}{x^q} = \left(1 + \frac{x_1}{x}\right) \cdots \left(1 + \frac{x_q}{x}\right).$$  \hspace{1cm} (2.9)

By using the following series expansion:

$$\ln \left(1 + \frac{z}{x}\right) = \sum_{j=1}^{q} \frac{(-1)^{j-1} x^j}{jx^j} + O\left(\frac{1}{x^{q+1}}\right)$$

for $|z| < |x|$ and $x \to \infty$, we obtain, as $x \to \infty$,

$$\ln \left(1 + \frac{c_1}{x} + \cdots + \frac{c_q}{x^q}\right) = \sum_{j=1}^{q} \frac{(-1)^{j-1} S_j}{jx^j} + O\left(\frac{1}{x^{q+1}}\right),$$  \hspace{1cm} (2.10)

where

$$S_j = x_1^j + \cdots + x_q^j \hspace{1cm} (1 \leq j \leq q).$$

We then find from (2.8) and (2.10) that

$$S_1 = \frac{\ln 2}{2} \quad \text{and} \quad S_j = \frac{(1 - 2^{1-j})B_j}{j-1} \hspace{1cm} (2 \leq j \leq q);$$  \hspace{1cm} (2.11)
that is,

\[
\begin{align*}
&x_1 + \cdots + x_q = \ln 2, \\
x_1^2 + \cdots + x_q^2 = \frac{B_2}{2}, \\
&\quad \cdots \cdots \\
x_1^q + \cdots + x_q^q = \frac{(1-2^{1-q})B_q}{q-1}.
\end{align*}
\] (2.12)

Let

\[
P_q(x) = x^q + b_1 x^{q-1} + \cdots + b_{q-1} x + b_q
\]
be a polynomial with zeros \(x_1, \ldots, x_q\) satisfying the system of equations (2.12). Then we have

\[
P_q(x) = (x - x_1) \cdots (x - x_q).
\] (2.13)

The Newton formulas (see, for example, [15] and the references therein) give the connection between the coefficients \(b_j\) and the power sums \(S_j\):

\[
S_j + S_j - 1 b_1 + S_j - 2 b_2 + \cdots + S_1 b_{j-1} + j b_j = 0 \quad (1 \leq j \leq q).
\]

It is known (see [15]) that the coefficients \(b_j\) can be expressed in terms of \(S_j\):

\[
b_j = \sum_{(k_1, k_2, \ldots, k_j) \in A_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1! k_2! \cdots k_j!} \left( \frac{S_1}{1} \right)^{k_1} \left( \frac{S_2}{2} \right)^{k_2} \cdots \left( \frac{S_j}{j} \right)^{k_j},
\] (2.14)

where the \(A_j\) \((j \in \mathbb{N})\) are given in (2.1). From (2.13) we therefore obtain

\[
\left( \frac{-1}{x^q} \right) P_q(-x) = \left( 1 + \frac{x_1}{x} \right) \cdots \left( 1 + \frac{x_q}{x} \right)
\]

so that

\[
1 - \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots + \frac{(-1)^q b_q}{x^q} = \left( 1 + \frac{x_1}{x} \right) \cdots \left( 1 + \frac{x_q}{x} \right).
\] (2.15)

We see from (2.9) and (2.15) that the coefficients \(c_j\) are then given by

\[
c_j = (-1)^j b_j = \sum_{(k_1, k_2, \ldots, k_j) \in A_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1! k_2! \cdots k_j!} \left( \frac{S_1}{1} \right)^{k_1} \left( \frac{S_2}{2} \right)^{k_2} \cdots \left( \frac{S_j}{j} \right)^{k_j},
\] (2.16)

where the \(S_j\) are specified in (2.11). This completes the proof.

Noting that

\[
2 \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = \left( \frac{\sqrt{\pi}}{\Gamma(n + \frac{1}{2})} \right)^{1/n}
\] (2.17)

holds, we obtain the following corollary.
Corollary 2.1. As \( n \to \infty \), we have

\[
2n \left( \sum_{j=0}^{q} \frac{c_j}{n^j} \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n} = e + O \left( \frac{1}{n^{q+1}} \right),
\]

(2.18)

where \( c_0 = 1 \) and the coefficients \( c_j \) \((1 \leq j \leq q)\) are given by (2.7).

Here we give explicit numerical values of the first few coefficients \( c_j \) by using the partition set (2.1) and the formula (2.7). This shows how easy it is to determine the coefficients \( c_j \) in (2.7). It is clear that

\[
c_1 = - \sum_{k_1=1} (-1)^{k_1} (S_1)_k = \frac{\ln 2}{2}.
\]

For \( k_1 + 2k_2 = 2 \), since \( p(2) = 2 \), the partition set \( A_2 \) in (2.1) is seen to have 2 elements:

\[
A_2 = \{(0,1), (2,0)\}.
\]

From (2.7) we have

\[
c_2 = \sum_{(k_1, k_2) \in A_2} (-1)^{k_1+k_2} (S_1)_{k_1} (S_2)_{k_2} = \frac{3(\ln 2)^2 - 1}{24}.
\]

For \( k_1 + 2k_2 + 3k_3 = 3 \), since \( p(3) = 3 \), the partition set \( A_3 \) in (2.1) contains 3 elements:

\[
A_3 = \{(0,0,1), (1,1,0), (3,0,0)\}
\]

and so we find from (2.7) that

\[
c_3 = - \sum_{(k_1, k_2, k_3) \in A_3} (-1)^{k_1+k_2+k_3} (S_1)_{k_1} (S_2)_{k_2} (S_3)_{k_3} = \frac{(\ln 2)^3 - \ln 2}{48},
\]

where \( 0^0 \) is interpreted as 1.

Likewise, the partition sets \( A_4 \) and \( A_5 \) have \( p(4) = 5 \) and \( p(5) = 7 \) elements, respectively, and so

\[
A_4 = \{(0,0,0,1), (1,0,1,0), (0,2,0,0), (2,1,0,0), (4,0,0,0)\};
\]

\[
A_5 = \{(0,0,0,0,1), (1,0,0,1,0), (0,1,1,0,0), (2,0,1,0,0),
\]

\[
(3,1,0,0,0), (5,0,0,0,0)\}
\]

which yields

\[
c_4 = \frac{19 - 30(\ln 2)^2 + 15(\ln 2)^4}{5760} \quad \text{and} \quad c_5 = \frac{(19 - 10(\ln 2)^2 + 3(\ln 2)^4) \ln 2}{11520}.
\]

This then produces the following asymptotic expansion:

\[
e \sim 2n \left( 1 + \frac{\ln 2}{2n} + \frac{3(\ln 2)^2 - 1}{24n^2} + \frac{(\ln 2)^3 - \ln 2}{48n^3} + \frac{19 - 30(\ln 2)^2 + 15(\ln 2)^4}{5760n^4}
\]

\[
+ \frac{19 - 10(\ln 2)^2 + 3(\ln 2)^4 \ln 2}{11520n^5} + \cdots \right) \left( \frac{2^n n!}{(2n)!} \right)^{1/n}
\]

(2.19)

as \( n \to \infty \).
3 Approximation formulas for \((1 + 1/x)^x\) and a Carleman-type inequality

Let \(a_n \geq 0\) for \(n \in \mathbb{N} := \{1, 2, \ldots\}\) and \(0 < \sum_{n=1}^{\infty} a_n < \infty\). Then

\[
\sum_{n=1}^{\infty} (a_1a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{3.1}
\]

The constant \(e\) is the best possible. The inequality (3.1) was presented in 1922 in [4] by the Swedish mathematician Torsten Carleman and it is now called Carleman’s inequality. Carleman discovered this inequality during his important work on quasi-analytical functions.

Carleman’s inequality (3.1) has been generalized by Hardy [17] (see also [18, p. 256]) as follows. If \(a_n \geq 0\), \(\lambda_n > 0\), \(\Lambda_n = \sum_{m=1}^{n} \lambda_m\) for \(n \in \mathbb{N}\), and \(0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty\), then

\[
\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1}a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{3.2}
\]

Note that inequality (3.2) is usually referred to as a Carleman-type inequality, or a weighted Carleman-type inequality. In his original paper [17], Hardy himself said that it was Pólya who pointed out this inequality to him. For information about the history of Carleman-type inequalities, see [21, 22, 24, 34].

3.1 Summary of previous results

In [5–7, 11, 12, 14, 25–27, 30, 31, 33, 39–44], some strengthened and generalized results of (3.1) and (3.2) have been given by estimating the weight coefficient \((1+1/n)^n\). For example, Mortici and Jang [33] proved that for \(0 < x \leq 1\),

\[
e \left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4 - \frac{959}{2304}x^5\right) \leq \left(1 + \frac{1}{n}\right)^n a_n, \tag{3.3}
\]

so that the following strengthened form of Carleman’s inequality can be derived directly from the right-hand side of (3.3) as

\[
\sum_{n=1}^{\infty} (a_1a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n} + \frac{11}{24n^2} - \frac{7}{16n^3} + \frac{2447}{5760n^4}\right) a_n. \tag{3.4}
\]

Brothers and Knox [3] (see also [8, 23]) derived, without a formula for the general term, the following expansion:

\[
\left(1 + \frac{1}{x}\right)^x = e \left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \cdots\right) \tag{3.5}
\]
for \( x < -1 \) or \( x \geq 1 \). With
\[
\left(1 + \frac{1}{x}\right)^x = e \sum_{j=0}^{\infty} \frac{a_j}{x^j}, \quad (x < -1 \text{ or } x \geq 1),
\]
(3.7)
Chen and Choi [8] gave an explicit formula for successively determining the coefficients \( a_j \) in the form
\[
a_0 = 1, \quad a_j = (-1)^j \sum_{(k_1, k_2, \ldots, k_j) \in \mathcal{A}_j} \frac{\left(\frac{1}{2}\right)^{k_1} \left(\frac{1}{3}\right)^{k_2} \cdots \left(\frac{1}{j+1}\right)^{k_j}}{k_1!k_2! \cdots k_j!},
\]
(3.8)
where the \( \mathcal{A}_j (j \in \mathbb{N}) \) are given in (2.1). The above result immediately shows that \((-1)^j a_j > 0\) so that (3.7) is an alternating series for positive \( x \). Recently, Chen and Paris [10] obtained a recurrence relation for \( \beta_j = (-1)^j a_j \) given by
\[
\beta_0 = 1 \quad \text{and} \quad \beta_j = \frac{1}{j} \sum_{k=1}^{j} k \beta_j - k \quad (j \geq 1).
\]
(3.9)
Use of (3.9) is easily seen to generate the values
\[
\beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{11}{24}, \quad \beta_3 = \frac{7}{16}, \quad \beta_4 = \frac{2447}{5760}, \quad \beta_5 = \frac{959}{2304}, \quad \beta_6 = \frac{238043}{580608}, \ldots,
\]
which are the same coefficients as in (3.6). The representation using a recursive algorithm for the coefficients \((-1)^j \beta_j = a_j \) in (3.9) is more practical for numerical evaluation than the expression in (3.8).

Chen and Paris [10] have given an integral representation for the coefficients \( \beta_j \) and have proved that the sequence \( \{\beta_j\}_{j=0}^{\infty} \) is monotonically decreasing. They thereby obtained the following double inequality [10, Theorem 2.1]:
\[
e \sum_{j=0}^{2m+1} \frac{(-1)^j \beta_j}{x^j} < \left(1 + \frac{1}{x}\right)^x < e \sum_{j=0}^{2m} \frac{(-1)^j \beta_j}{x^j} \quad (x \geq 1),
\]
(3.10)
which develops the double inequality (3.3) to produce a general result. As an application of (3.10), Chen and Paris [10, Theorem 3.1] have given a generalized Carleman-type inequality.

In 2001 Yang [43] conjectured, then Yang [44], Gylleberg and Yan [16], Chen [5], Lü et al. [27], and Hu and Mortici [20] proved that if the following equality holds:
\[
\left(1 + \frac{1}{x}\right)^x = e \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(1 + x)^k}\right)
\]
(3.11)
for \( x > 0 \), then \( b_k > 0 \) for \( k \in \mathbb{N} \). In fact, Yang [44], Gylleberg and Yan [16], and Chen [5] presented the following recurrence relation for determining the coefficients \( b_k \) in (3.11):
\[
b_1 = \frac{1}{2}, \quad b_{n+1} = \frac{1}{n+1} \left(\frac{1}{n+2} - \sum_{j=1}^{n} \frac{b_j}{n+2-j}\right) \quad (n \geq 1),
\]
(3.12)
and then proved \( b_k > 0 \) for \( k \in \mathbb{N} \); see also Lü et al. [27]. Hu and Mortici [20] used an argument of Alzer and Berg [2] to derive an integral representation for \( b_k \), and then obtained some new properties of \( b_k \), including \( b_k > 0 \) for \( k \in \mathbb{N} \). We remark that the recurrence relation of the coefficients \( b_k \) given in [19, Lemma 2.2] is not correct.
Remark 3.1. We give here an explicit formula for determining the coefficients $b_k$ in (3.11):

$$b_j = -\sum_{(k_1, k_2, \ldots, k_j) \in A_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left( \frac{1}{1 \cdot 2} \right)^{k_1} \left( \frac{1}{2 \cdot 3} \right)^{k_2} \cdots \left( \frac{1}{j(j+1)} \right)^{k_j},$$  \hspace{1cm} (3.13)

where the $A_j$ ($j \in \mathbb{N}$) are given in (2.1).

Noting that $b_k > 0$ for $k \in \mathbb{N}$ in (3.11), it follows from (3.11) that

$$\left( 1 + \frac{1}{x} \right)^x < e \left( 1 - \sum_{k=1}^{m} \frac{b_k}{(1 + x)^k} \right)$$  \hspace{1cm} (3.14)

for $x > 0$ and $m \in \mathbb{N}$. As an application of (3.14), inequalities (3.2) and (3.1) were strengthened by Yang [44, Corollaries 2 and 3].

In the final part of his paper, Yang [43] remarked that in order to obtain better results, the right-hand side of (3.11) could be replaced by $e[1 - \sum_{n=1}^{\infty} (d_n/(x + \varepsilon)^n)]$, where $\varepsilon \in (0, 1]$ and $d_n = d_n(\varepsilon)$, but information about the values of $\varepsilon$ are not provided. In fact, Xie and Zhong [39] proved in 2000 that $x \geq 1$,

$$e \left( 1 - \frac{7}{14x + 12} \right) < \left( 1 + \frac{1}{x} \right)^x < e \left( 1 - \frac{6}{12x + 11} \right),$$  \hspace{1cm} (3.15)

and then applied it to obtain an improvement of (3.2) as follows: if $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0$ ($n \in \mathbb{N}$) and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_n} a_2^{\lambda_n} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1}{\Lambda_n/\lambda_n + \frac{11}{12}} \right) \lambda_n a_n.$$

(3.16)

Recently, Mortici and Hu [32] gave a formula for determining the coefficients $d_k$ such that

$$\left( 1 + \frac{1}{n} \right)^n = e \left( 1 - \sum_{k=1}^{\infty} \frac{d_k}{(1 + n)^k} \right)$$  \hspace{1cm} (3.17)

which is better than (3.11), since by truncation after $k \geq 3$ terms of series (3.11), the last term is of order $n^{-(k-1)}$, while the last term of series (3.17) truncated after $k$ terms is of order $n^{-k}$. For the same reason, the formula (3.17) is better than (3.6).

Let

$$\left( 1 + \frac{1}{x} \right)^x = e \left( 1 - \sum_{k=1}^{\infty} \frac{b_k}{(1 + x)^k} \right) = e \left( 1 - \sum_{k=1}^{\infty} \frac{d_k}{(1 + x)^k} \right),$$

$$\sigma_m(x) = \sum_{k=1}^{m} \frac{b_k}{(1 + x)^k} \quad \text{and} \quad S_m(x) = \sum_{k=1}^{m} \frac{d_k}{(1 + x)^k}.$$  \hspace{1cm} (3.18)

Then Ren and Li [36] proved that (i) if $m \geq 6$ is even, we have $S_m(x) > \sigma_m(x)$ for all $x > 0$ and (ii) if $m \geq 7$ is odd, we have $S_m(x) > \sigma_m(x)$ for all $x > 1$. This provides an intuitive explanation for the main result in Mortici and Hu [32].

Recently, You et al. [45] provided continued fraction inequalities related to $(1 + 1/x)^x$, which can be used to refine the inequalities (3.1) and (3.2).
3.2 A new form of approximation for \((1 + 1/x)^x\)

Using the Maple software, we find

\[
\left(1 + \frac{1}{x}\right)^x \sim e \left(1 - \frac{\frac{1}{2}}{x + \frac{11}{12}} - \frac{\frac{5}{288}}{(x + \frac{343}{450})^3} - \frac{\frac{41683}{7552000}}{(x + \frac{568100391}{787808700})^5} - \cdots \right)
\]  

(3.18)
as \(x \to \infty\). This led us to pose the following problem: Find the constants \(\lambda_\ell\) and \(\mu_\ell\) such that

\[
\left(1 + \frac{1}{x}\right)^x \sim e \left(1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(x + \mu_\ell)^{2\ell - 1}}\right)
\]
as \(x \to \infty\). In this section we solve this problem. Thus, we would appear to obtain an odd-type asymptotic expansion for \((1 + 1/x)^x\). From a computational viewpoint, (3.18) is an improvement on the formulas (3.6), (3.11) and (3.17).

**Theorem 3.1.** As \(x \to \infty\), we have

\[
\left(1 + \frac{1}{x}\right)^x \sim e \left(1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(x + \mu_\ell)^{2\ell - 1}}\right),
\]

(3.19)

where the constants \(\lambda_\ell\) and \(\mu_\ell\) are given by the pair of recurrence relations

\[
\lambda_\ell = a_{2\ell - 1} - \sum_{k=1}^{\ell-1} \lambda_k \mu_{2\ell - 2k} \left(\frac{2\ell - 2}{2\ell - 2k}\right) \quad (\ell \geq 2)
\]

(3.20)

and

\[
\mu_\ell = -\frac{1}{(2\ell - 1)\lambda_\ell} \left\{ a_{2\ell} + \sum_{k=1}^{\ell-1} \lambda_k \mu_{2\ell - 2k + 1} \left(\frac{2\ell - 1}{2\ell - 2k + 1}\right) \right\} \quad (\ell \geq 2),
\]

(3.21)

with \(\lambda_1 = -\frac{1}{2}\) and \(\mu_1 = \frac{11}{12}\). Here \(a_j\) are given in (3.7).

**Proof.** We first express (3.19) in the form

\[
e^{-1} \left(1 + \frac{1}{x}\right)^x - 1 \sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}} \left(1 + \frac{\mu_j}{x}\right)^{2j+1}.
\]

1Using the Maple software, formula (3.18) is given in the appendix.
Direct computation yields
\[
\sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}} \left( 1 + \frac{\mu_j}{x} \right)^{-2j+1} = \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}} \sum_{k=0}^{\infty} \binom{-2j+1}{k} \frac{\mu_j^k}{x^k}
\]
\[
= \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-2}{k} \frac{\mu_j^k}{x^k}
\]
\[
= \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \lambda_{j+1} \mu_{j+1} (-1)^{j-k-1} \binom{j+k-1}{j-k-1} \frac{1}{x^{j+k}}
\]
\[
= \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j+1} \lambda_k \mu_k^{-2k+1} (-1)^{j-k} \binom{j-1}{j-2k+1} \right\} \frac{1}{x^j}.
\]

We then obtain
\[
e^{-1} \left( 1 + \frac{1}{x} \right)^x - 1 \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j+1} \lambda_k \mu_k^{-2k+1} (-1)^{j-k} \binom{j-1}{j-2k+1} \right\} \frac{1}{x^j}.
\]

(3.22)

On the other hand, it follows from (3.7) that
\[
e^{-1} \left( 1 + \frac{1}{x} \right)^x - 1 = \sum_{j=1}^{\infty} a_j \frac{1}{x^j},
\]

(3.23)

where \( a_j \) are given in (3.8). Equating coefficients of the term \( x^{-j} \) on the right-hand sides of (3.22) and (3.23), we obtain
\[
a_j = \sum_{k=1}^{j+1} \lambda_k \mu_k^{-2k+1} (-1)^{j-k} \binom{j-1}{j-2k+1} \quad (j \in \mathbb{N}).
\]

(3.24)

Setting \( j = 2\ell - 1 \) and \( j = 2\ell \) in (3.24), respectively, we find
\[
a_{2\ell-1} = \sum_{k=1}^{\ell} \lambda_k \mu_k^{-2\ell+2k} \left( \frac{2\ell}{2\ell+2k} \right)
\]

(3.25)

and
\[
a_{2\ell} = - \sum_{k=1}^{\ell+1} \lambda_k \mu_k^{-2\ell+2k+1} \left( \frac{2\ell-1}{2\ell+2k+1} \right)
\]
\[
- \sum_{k=1}^{\ell} \lambda_k \mu_k^{-2\ell+2k+1} \left( \frac{2\ell}{2\ell+2k+1} \right) - \lambda_{\ell+1} \mu_{\ell+1}^{-2\ell+1} \left( -1 \right)
\]
\[
= - \sum_{k=1}^{\ell} \lambda_k \mu_k^{-2\ell+2k+1} \left( \frac{2\ell-1}{2\ell+2k+1} \right)
\]

(3.26)
From (3.25) and (3.26) we obtain for \( \ell = 1 \),
\[
\lambda_1 = a_1 = -\frac{1}{2} \quad \text{and} \quad \mu_1 = -\frac{a_2}{\lambda_1} = \frac{11}{12},
\]
and for \( \ell \geq 2 \) we have
\[
a_{2\ell-1} = \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \left( \frac{2\ell-2}{2\ell-2k} \right) + \lambda_1
\]
and
\[
a_{2\ell} = -\sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1} \left( \frac{2\ell-1}{2\ell-2k+1} \right) - (2\ell-1)\lambda_1 \mu_1.
\]
We then obtain the recurrence relations (3.20) and (3.21). The proof is complete.

We give explicit numerical values of the first few constants \( \lambda_\ell \) and \( \mu_\ell \) by using the formulas (3.20) and (3.21). This demonstrates the ease with which the constants \( \lambda_\ell \) and \( \mu_\ell \) in (3.19) can be determined.

\[
\lambda_1 = -\frac{1}{2}, \quad \mu_1 = \frac{11}{12},
\]
\[
\lambda_2 = a_3 - \lambda_1 \mu_1^2 = -\frac{7}{16} - \left( -\frac{1}{2} \right) \cdot \left( \frac{11}{12} \right)^2 = -\frac{5}{288},
\]
\[
\mu_2 = -\frac{a_4 + \lambda_1 \mu_1^3}{3 \lambda_2} = -\frac{2447}{5760} + \left( -\frac{1}{2} \right) \cdot \left( \frac{11}{12} \right)^3 = \frac{343}{450},
\]
\[
\lambda_3 = a_5 - \lambda_1 \mu_1^4 - 6 \lambda_2 \mu_2^2 = \frac{959}{2304} - \left( -\frac{1}{2} \right) \cdot \left( \frac{11}{12} \right)^4 - 6 \cdot \left( -\frac{5}{288} \right) \cdot \left( \frac{343}{450} \right)^2 = -\frac{41683}{15552000},
\]
\[
\mu_3 = -\frac{a_6 + \lambda_1 \mu_1^5 + 10 \lambda_2 \mu_2^3}{5 \lambda_3} = \frac{238043}{580608} + \left( -\frac{1}{2} \right) \cdot \left( \frac{11}{12} \right)^5 + 10 \cdot \left( -\frac{5}{288} \right) \cdot \left( \frac{343}{450} \right)^3 = \frac{558100391}{787808700}.
\]

We note that the values of \( \lambda_\ell \) and \( \mu_\ell \) (for \( \ell = 1, 2, 3 \)) above are equal to the constants appearing in (3.18).

**Remark 3.2.** By using the Maple software, we can show that for \( x > 0 \),
\[
\left( 1 + \frac{1}{x} \right)^x < e \left( 1 - \frac{1}{x + \frac{11}{12}} - \frac{5}{x + \frac{343}{450}} - \frac{41683}{x + \frac{558100391}{787808700}} \right), \quad (3.27)
\]
We omit the proof.
By virtue of the proof given in [42] and the inequality (3.27), we have the Carleman-type inequality

\[
\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_n^{\lambda_n})^{1/\Lambda_n} < \sum_{n=1}^{\infty} \left(1 + \frac{1}{\Lambda_n/\lambda_n}\right)^{\Lambda_n/\lambda_n} \lambda_n a_n,
\]

\[
< e \sum_{n=1}^{\infty} \left(1 - \frac{\frac{5}{288}}{\left(\Lambda_n/\lambda_n\right)^3} - \frac{\frac{41683}{15552000}}{\left(\Lambda_n/\lambda_n\right)^{\frac{5}{2}}}ight) \lambda_n a_n,
\]

which is an improvement on the inequality (3.16).

Finally, we propose the following conjecture.

**Conjecture 3.1.** For all \( \ell \in \mathbb{N} \), we have

\[
\lambda_\ell < 0 \quad \text{and} \quad \mu_\ell > 0.
\]

Further, we have the inequality

\[
\left(1 + \frac{1}{x}\right)^x < e \left(1 + \sum_{\ell=1}^{m} \frac{\lambda_\ell}{x + \mu_\ell} 2^{x-1}\right)
\]

for \( x > 0 \) and \( m \in \mathbb{N} \).

Appendix: A derivation of formula (3.18)

Define the function \( F(x) \) by

\[
F(x) = \left(1 + \frac{1}{x}\right)^x - e \left(1 + \frac{\lambda_1}{x + \mu_1}\right).
\]

We are interested in finding the values of the parameters \( \lambda_1 \) and \( \mu_1 \) such that \( F(x) \) converges as fast as possible to zero, as \( x \to \infty \). This provides the best approximation of the form:

\[
\left(1 + \frac{1}{x}\right)^x \approx e \left(1 + \frac{\lambda_1}{x + \mu_1}\right).
\]

Using the Maple software, we find, as \( x \to \infty \),

\[
F(x) = -e\left(1 + 2\lambda_1\right) + \frac{e(24\lambda_1\mu_1 + 11)}{24x^2} - \frac{e(16\lambda_1^2 + 7)}{16x^3} + O\left(\frac{1}{x^4}\right).
\]

The two parameters \( \lambda_1 \) and \( \mu_1 \), which produce the fastest convergence of the function \( F(x) \), are given by

\[
\begin{cases}
1 + 2\lambda_1 = 0 \\
24\lambda_1\mu_1 + 11 = 0,
\end{cases}
\]

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namely, if
\[ \lambda_1 = -\frac{1}{2}, \quad \mu_1 = \frac{11}{12}. \]
We then obtain, as \( x \to \infty \),
\[ (1 + \frac{1}{x})^x = e\left(1 - \frac{1}{x + \frac{11}{12}} + O\left(\frac{1}{x^3}\right)\right). \tag{3.31} \]
In view of (3.31), we define the function \( G(x) \) by
\[ G(x) = \left(1 + \frac{1}{x}\right)^x - e\left(1 - \frac{1}{x + \frac{11}{12}} + \frac{\lambda_2}{(x + \mu_2)^3}\right). \]
Using the Maple software, we find, as \( x \to \infty \),
\[ G(x) = -\frac{e(5 + 288\lambda_2)}{288x^3} + \frac{e(343 + 25920\lambda_2\mu_2)}{8640x^4} - \frac{e(2621 + 248832\lambda_2\mu_2^2)}{41472x^5} + O\left(\frac{1}{x^6}\right). \]
For \( \lambda_2 = -\frac{5}{288} \) and \( \mu_2 = \frac{343}{450} \), we obtain, as \( x \to \infty \),
\[ (1 + \frac{1}{x})^x = e\left(1 - \frac{1}{x + \frac{11}{12}} - \frac{5}{288} + O\left(\frac{1}{x^5}\right)\right). \tag{3.32} \]
In view of (3.32), we define the function \( H(x) \) by
\[ H(x) = \left(1 + \frac{1}{x}\right)^x - e\left(1 - \frac{1}{x + \frac{11}{12}} - \frac{5}{288} + \frac{\lambda_3}{(x + \mu_3)^5}\right). \]
Using the Maple software, we find, as \( x \to \infty \),
\[ H(x) = -\frac{e(41683 + 15552000\lambda_3)}{15552000x^5} + \frac{e(558100391 + 2939328000000\lambda_3\mu_3)}{587865600000x^6} - \frac{e(52111420409 + 37791360000000\lambda_3\mu_3^2)}{2519424000000x^7} + O\left(\frac{1}{x^8}\right). \]
For \( \lambda_3 = -\frac{41683}{15552000} \) and \( \mu_3 = \frac{558100391}{787808700} \), we obtain, as \( x \to \infty \),
\[ (1 + \frac{1}{x})^x = e\left(1 - \frac{1}{x + \frac{11}{12}} - \frac{5}{288} + \frac{41683}{15552000} + O\left(\frac{1}{x^7}\right)\right). \]

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