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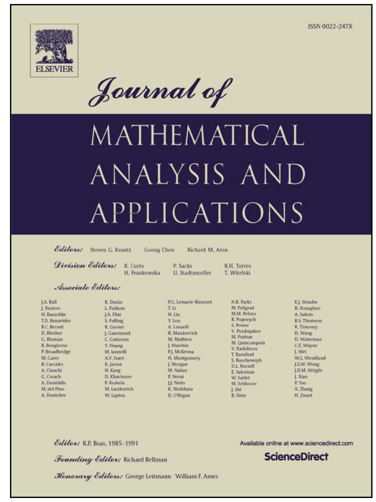

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Approximation formulas for the constant e and an improvement to a Carleman-type inequality

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Abstract. We give an explicit formula for the determination of the coefficients c_j appearing in the expansion

$$x\left(1+\sum_{j=1}^{q}\frac{c_j}{x^j}\right)\left(\frac{\sqrt{\pi}}{\Gamma\left(x+\frac{1}{2}\right)}\right)^{1/x}=e+O\left(\frac{1}{x^{q+1}}\right)$$

for $x\to\infty$ and $q\in\mathbb{N}:=\{1,2,\ldots\}$. We also derive a pair of recurrence relations for the determination of the constants λ_ℓ and μ_ℓ in the expansion

$$\left(1+\frac{1}{x}\right)^x\sim e\left(1+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)$$

as $x\to\infty$. Based on this expansion, we establish an inequality for $(1+1/x)^x$. As an application, we give an improvement to a Carleman-type inequality.

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1 Introduction

The constant e can be defined by the limit

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

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With the possible exception of π, e is the most important constant in mathematics since it appears in myriad mathematical contexts involving limits and derivatives. Joost Bürgi seems to have been the first to formulate an approximation to e around 1620, obtaining three-decimal-place accuracy (see [13, p. 31], [23], and [29, pp. 26–27]).

Theorems 1.1 and 1.2 below were proved by Chen and Mortici [9].

Theorem 1.1. *For all* $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$,

$$2\left(n+\alpha\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} < e \le 2\left(n+\beta\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} \tag{1.1}$$

with the best possible constants

$$\alpha = \frac{\ln 2}{2} = 0.34657\ldots \quad \textit{and} \quad \beta = \frac{e}{2} - 1 = 0.35914\ldots.$$

Theorem 1.2. *Let* $(v_n)_{n\in\mathbb{N}}$ *be defined by*

$$v_n = 2\left(n + \frac{\ln 2}{2} + \frac{a}{n} + \frac{b}{n^2}\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n}. \tag{1.2}$$

Then, for

$$a = \frac{3(\ln 2)^2 - 1}{24}, \quad b = \frac{(\ln 2)^3 - \ln 2}{48},$$

we have

$$\lim_{n\to\infty} n^4(v_n - e) = -\frac{e\big(19 - 30(\ln 2)^2 + 15(\ln 2)^4\big)}{5760}.$$

The speed of convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ *is* n^{-4}.

By using the Maple software, we find, as $n \to \infty$,

$$2n\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = e + O\left(\frac{1}{n}\right), \tag{1.3}$$

$$2n\left(1 + \frac{\ln 2}{2n}\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = e + O\left(\frac{1}{n^2}\right), \tag{1.4}$$

$$2n\left(1 + \frac{\ln 2}{2n} + \frac{3(\ln 2)^2 - 1}{24n^2}\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = e + O\left(\frac{1}{n^3}\right) \tag{1.5}$$

and

$$2n\left(1 + \frac{\ln 2}{2n} + \frac{3(\ln 2)^2 - 1}{24n^2} + \frac{(\ln 2)^3 - \ln 2}{48n^3}\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = e + O\left(\frac{1}{n^4}\right). \tag{1.6}$$

Motivated by (1.3)-(1.6), we first establish a general approximation formula for e (given in Theorem 2.1, by mainly using the partition function. From this result, we give an explicit formula for the coefficients c_j $(1 \le j \le q)$ such that

$$2n\left(1 + \sum_{j=1}^{q}\frac{c_j}{n^j}\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = e + O\left(\frac{1}{n^{q+1}}\right) \tag{1.7}$$

for $n \to \infty$ and $q \in \mathbb{N}$, which contains the formulas (1.3)-(1.6) as special cases.

The second aim of the paper is to derive a pair of recurrence relations for the determination of the constants λ_ℓ and μ_ℓ in the expansion

$$\left(1+\frac{1}{x}\right)^x \sim e\left(1+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)$$

as $x \to \infty$ (given in Theorem 3.1). Based on this expansion, we establish an inequality for $(1+1/x)^x$ and, as an application, we give an improvement to a Carleman-type inequality (Remark 3.2).

2 The general form of the coefficients c_j in (1.7)

For our later use, we introduce the following set of partitions of an integer $n \in \mathbb{N}$:

$$\mathcal{A}_n := \{(k_1,\, k_2,\, \ldots,\, k_n) \in \mathbb{N}_0^n \,:\, k_1 + 2k_2 + \cdots + nk_n = n\}\,. \tag{2.1}$$

In number theory, the partition function $p(n)$ represents the number of possible partitions of $n \in \mathbb{N}$; that is, the number of distinct ways of representing n as a sum of natural numbers (with order irrelevant). By convention $p(0) = 1$ and $p(n) = 0$ for n negative integers. For more information on the partition function $p(n)$, see [38] and the references therein. The first few values of the partition function $p(n)$ are (starting with $p(0) = 1$) (see [37]):

$$1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11,\ 15,\ 22,\ 30,\ 42,\ \ldots .$$

It is easy to see that the cardinality of the set $\mathcal{A}_n$ is equal to the partition function $p(n)$.

The following results are needed in our present investigation. The logarithm of the gamma function has the asymptotic expansion (see [28, p. 32]):

$$\ln\Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}\frac{1}{x^n} \tag{2.2}$$

as $x \to \infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x-1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}. \tag{2.3}$$

Note that the Bernoulli numbers B_n are defined by $B_n := B_n(0)$ in (2.3).

Taking $t = \frac{1}{2}$ in (2.2), we have

$$\ln\Gamma\left(x+\frac{1}{2}\right) \sim x\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(\frac{1}{2})}{n(n+1)}\frac{1}{x^n} \tag{2.4}$$

as $x \to \infty$. Noting that

$$B_n\left(\frac{1}{2}\right) = (2^{1-n}-1)B_n \quad \text{for} \quad n \in \mathbb{N}_0$$

(see [1, p. 805, 23.1.21]), we find from (2.4) that

$$1+\frac{1}{x}\ln\Gamma\left(x+\frac{1}{2}\right) - \ln x - \frac{1}{2x}\ln(\pi) = \frac{\ln 2}{2x} + \sum_{j=2}^{q}\frac{(-1)^{j-1}(1-2^{1-j})B_j}{(j-1)j}\frac{1}{x^j} + O\left(\frac{1}{x^{q+1}}\right) \tag{2.5}$$

as $x \to \infty$.

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Theorem 2.1. *The following approximation formula for the constant* e *holds true:*

$$x\left(1+\sum_{j=1}^{q}\frac{c_j}{x^j}\right)\left(\frac{\sqrt{\pi}}{\Gamma\left(x+\frac{1}{2}\right)}\right)^{1/x}=e+O\left(\frac{1}{x^{q+1}}\right) \tag{2.6}$$

for $x\to\infty$ *and* $q\in\mathbb{N}$*, with the coefficients* c_j $(1\leq j\leq q)$ *given by*

$$c_j=(-1)^j\sum_{(k_1,\,k_2,\,\ldots,\,k_j)\in\mathcal{A}_j}\frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\cdots\left(\frac{S_j}{j}\right)^{k_j}, \tag{2.7}$$

where the $\mathcal{A}_j$ $(for\;\; j\in\mathbb{N})$ *are given in* (2.1)*,*

$$S_1=\frac{\ln 2}{2},\quad S_j=\frac{(1-2^{1-j})B_j}{j-1}\quad (2\leq j\leq q),$$

and B_n *are the Bernoulli numbers.*

Proof. To determine the coefficients c_j $(1\leq j\leq q)$, we first express (2.6) in the form

$$\ln\left(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\cdots+\frac{c_q}{x^q}\right)=\frac{\ln 2}{2x}+\sum_{j=2}^{q}\frac{(-1)^{j-1}(1-2^{1-j})B_j}{(j-1)j}\frac{1}{x^j}+O\left(\frac{1}{x^{q+1}}\right) \tag{2.8}$$

as $x\to\infty$, upon making use of (2.5). From the fundamental theorem of algebra, we see that there exist unique complex numbers $x_1,\,\ldots,\,x_q$ such that

$$1+\frac{c_1}{x}+\cdots+\frac{c_q}{x^q}=\left(1+\frac{x_1}{x}\right)\cdots\left(1+\frac{x_q}{x}\right). \tag{2.9}$$

By using the following series expansion:

$$\ln\left(1+\frac{z}{x}\right)=\sum_{j=1}^{q}\frac{(-1)^{j-1}z^j}{jx^j}+O\left(\frac{1}{x^{q+1}}\right)$$

for $|z|<|x|$ and $x\to\infty$, we obtain, as $x\to\infty$,

$$\ln\left(1+\frac{c_1}{x}+\cdots+\frac{c_q}{x^q}\right)=\sum_{j=1}^{q}\frac{(-1)^{j-1}S_j}{jx^j}+O\left(\frac{1}{x^{q+1}}\right), \tag{2.10}$$

where

$$S_j=x_1^j+\cdots+x_q^j\qquad(1\leq j\leq q).$$

We then find from (2.8) and (2.10) that

$$S_1=\frac{\ln 2}{2}\quad\text{and}\quad S_j=\frac{(1-2^{1-j})B_j}{j-1}\qquad(2\leq j\leq q); \tag{2.11}$$

that is,

$$\begin{cases} x_1 + \cdots + x_q = \frac{\ln 2}{2}, \\ x_1^2 + \cdots + x_q^2 = \frac{B_2}{2}, \\ \cdots\cdots \\ x_1^q + \cdots + x_q^q = \frac{(1-2^{1-q})B_q}{q-1}. \end{cases} \tag{2.12}$$

Let

$$P_q(x) = x^q + b_1 x^{q-1} + \cdots + b_{q-1}x + b_q$$

be a polynomial with zeros $x_1, \ldots, x_q$ satisfying the system of equations (2.12). Then we have

$$P_q(x) = (x - x_1)\cdots(x - x_q). \tag{2.13}$$

The Newton formulas (see, for example, [15] and the references therein) give the connection between the coefficients b_j and the power sums S_j:

$$S_j + S_{j-1}b_1 + S_{j-2}b_2 + \cdots + S_1 b_{j-1} + j b_j = 0 \qquad (1 \leq j \leq q).$$

It is known (see [15]) that the coefficients b_j can be expressed in terms of S_j:

$$b_j = \sum_{(k_1,\, k_2,\, \ldots,\, k_j)\in\mathcal{A}_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\cdots\left(\frac{S_j}{j}\right)^{k_j}, \tag{2.14}$$

where the $\mathcal{A}_j$ $(j \in \mathbb{N})$ are given in (2.1).

From (2.13) we therefore obtain

$$\frac{(-1)^q}{x^q}P_q(-x) = \left(1 + \frac{x_1}{x}\right)\cdots\left(1 + \frac{x_q}{x}\right)$$

so that

$$1 - \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots + \frac{(-1)^q b_q}{x^q} = \left(1 + \frac{x_1}{x}\right)\cdots\left(1 + \frac{x_q}{x}\right). \tag{2.15}$$

We see from (2.9) and (2.15) that the coefficients c_j are then given by

$$\begin{aligned} c_j &= (-1)^j\, b_j \\ &= (-1)^j \sum_{(k_1,\, k_2,\, \ldots,\, k_j)\in\mathcal{A}_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\cdots\left(\frac{S_j}{j}\right)^{k_j}, \end{aligned} \tag{2.16}$$

where the S_j are specified in (2.11). This completes the proof. □

Noting that

$$2\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = \left(\frac{\sqrt{\pi}}{\Gamma\left(n + \frac{1}{2}\right)}\right)^{1/n} \tag{2.17}$$

holds, we obtain the following corollary.

Corollary 2.1. As $n \to \infty$, we have

$$2n\left(\sum_{j=0}^{q}\frac{c_j}{n^j}\right)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} = e + O\left(\frac{1}{n^{q+1}}\right), \tag{2.18}$$

where $c_0 = 1$ and the coefficients c_j $(1 \le j \le q)$ are given by (2.7).

Here we give explicit numerical values of the first few coefficients c_j by using the partition set (2.1) and the formula (2.7). This shows how easy it is to determine the coefficients c_j in (2.7). It is clear that

$$c_1 = -\sum_{k_1=1}\frac{(-1)^{k_1}}{k_1!}\left(\frac{S_1}{1}\right)^{k_1} = \frac{\ln 2}{2}.$$

For $k_1 + 2k_2 = 2$, since $p(2) = 2$, the partition set $\mathcal{A}_2$ in (2.1) is seen to have 2 elements:

$$\mathcal{A}_2 = \{(0,1),\ (2,0)\}.$$

From (2.7) we have

$$c_2 = \sum_{(k_1,\,k_2)\in\mathcal{A}_2}\frac{(-1)^{k_1+k_2}}{k_1!k_2!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2} = \frac{3(\ln 2)^2 - 1}{24}.$$

For $k_1 + 2k_2 + 3k_3 = 3$, since $p(3) = 3$, the partition set $\mathcal{A}_3$ in (2.1) contains 3 elements:

$$\mathcal{A}_3 = \{(0,0,1),\ (1,1,0),\ (3,0,0)\}$$

and so we find from (2.7) that

$$c_3 = -\sum_{(k_1,\,k_2,\,k_3)\in\mathcal{A}_3}\frac{(-1)^{k_1+k_2+k_3}}{k_1!k_2!k_3!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\left(\frac{S_3}{3}\right)^{k_3} = \frac{(\ln 2)^3 - \ln 2}{48},$$

where 0^0 is interpreted as 1.

Likewise, the partition sets $\mathcal{A}_4$ and $\mathcal{A}_5$ have $p(4) = 5$ and $p(5) = 7$ elements, respectively, and so

$$\begin{aligned}\mathcal{A}_4 &= \{(0,0,0,1),\ (1,0,1,0),\ (0,2,0,0),\ (2,1,0,0),\ (4,0,0,0)\};\\ \mathcal{A}_5 &= \{(0,0,0,0,1),\ (1,0,0,1,0),\ (0,1,1,0,0),\ (2,0,1,0,0),\\ &\quad (1,2,0,0,0),\ (3,1,0,0,0),\ (5,0,0,0,0)\}\end{aligned}$$

which yields

$$c_4 = \frac{19 - 30(\ln 2)^2 + 15(\ln 2)^4}{5760} \quad \text{and} \quad c_5 = \frac{\left(19 - 10(\ln 2)^2 + 3(\ln 2)^4\right)\ln 2}{11520}.$$

This then produces the following asymptotic expansion:

$$\begin{aligned} e \sim 2n\Bigg(1 &+ \frac{\ln 2}{2n} + \frac{3(\ln 2)^2 - 1}{24n^2} + \frac{(\ln 2)^3 - \ln 2}{48n^3} + \frac{19 - 30(\ln 2)^2 + 15(\ln 2)^4}{5760n^4}\\ &+ \frac{\left(19 - 10(\ln 2)^2 + 3(\ln 2)^4\right)\ln 2}{11520n^5} + \cdots\Bigg)\left(\frac{2^n n!}{(2n)!}\right)^{1/n} \end{aligned} \tag{2.19}$$

as $n \to \infty$.

3 Approximation formulas for $(1+1/x)^x$ and a Carleman-type inequality

Let $a_n \geq 0$ for $n \in \mathbb{N} := \{1, 2, \ldots\}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{3.1}$$

The constant e is the best possible. The inequality (3.1) was presented in 1922 in [4] by the Swedish mathematician Torsten Carleman and it is now called Carleman's inequality. Carleman discovered this inequality during his important work on quasi-analytical functions.

Carleman's inequality (3.1) has been generalized by Hardy [17] (see also [18, p. 256]) as follows. If $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{3.2}$$

Note that inequality (3.2) is usually referred to as a Carleman-type inequality, or a weighted Carleman-type inequality. In his original paper [17], Hardy himself said that it was Pólya who pointed out this inequality to him. For information about the history of Carleman-type inequalities, see [21, 22, 24, 34].

3.1 Summary of previous results

In [5–7, 11, 12, 14, 25–27, 30, 31, 33, 39–44], some strengthened and generalized results of (3.1) and (3.2) have been given by estimating the weight coefficient $(1+1/n)^n$. For example, Mortici and Jang [33] proved that for $0 < x \leq 1$,

$$e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4 - \frac{959}{2304}x^5\right) < (1+x)^{1/x} < e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4\right). \tag{3.3}$$

According to Pólya's proof of (3.1) in [35],

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n a_n, \tag{3.4}$$

so that the following strengthened form of Carleman's inequality can be derived directly from the right-hand side of (3.3) as

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n} + \frac{11}{24n^2} - \frac{7}{16n^3} + \frac{2447}{5760n^4}\right) a_n. \tag{3.5}$$

Brothers and Knox [3] (see also [8, 23]) derived, without a formula for the general term, the following expansion:

$$\left(1 + \frac{1}{x}\right)^x = e\left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \cdots\right) \tag{3.6}$$

for $x < -1$ or $x \geq 1$. With

$$\left(1+\frac{1}{x}\right)^x = e\sum_{j=0}^{\infty}\frac{a_j}{x^j}, \qquad (x < -1 \quad \text{or} \quad x \geq 1), \tag{3.7}$$

Chen and Choi [8] gave an explicit formula for successively determining the coefficients a_j in the form

$$a_0 = 1, \quad a_j = (-1)^j \sum_{(k_1,\, k_2,\, \ldots,\, k_j)\in\mathcal{A}_j} \frac{\left(\frac{1}{2}\right)^{k_1}\left(\frac{1}{3}\right)^{k_2}\cdots\left(\frac{1}{j+1}\right)^{k_j}}{k_1!k_2!\cdots k_j!}, \tag{3.8}$$

where the $\mathcal{A}_j$ ($j \in \mathbb{N}$) are given in (2.1). The above result immediately shows that $(-1)^j a_j > 0$ so that (3.7) is an alternating series for positive x. Recently, Chen and Paris [10] obtained a recurrence relation for $\beta_j = (-1)^j a_j$ given by

$$\beta_0 = 1 \quad \text{and} \quad \beta_j = \frac{1}{j}\sum_{k=1}^{j}\frac{k}{k+1}\beta_{j-k} \qquad (j \geq 1). \tag{3.9}$$

Use of (3.9) is easily seen to generate the values

$$\beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{11}{24}, \quad \beta_3 = \frac{7}{16}, \quad \beta_4 = \frac{2447}{5760}, \quad \beta_5 = \frac{959}{2304}, \quad \beta_6 = \frac{238043}{580608}, \ \ldots,$$

which are the same coefficients as in (3.6). The representation using a recursive algorithm for the coefficients $(-1)^j\beta_j = a_j$ in (3.9) is more practical for numerical evaluation than the expression in (3.8).

Chen and Paris [10] have given an integral representation for the coefficients β_j and have proved that the sequence $\{\beta_j\}_{j=0}^{\infty}$ is monotonically decreasing. They thereby obtained the following double inequality [10, Theorem 2.1]:

$$e\sum_{j=0}^{2m+1}\frac{(-1)^j\beta_j}{x^j} < \left(1+\frac{1}{x}\right)^x < e\sum_{j=0}^{2m}\frac{(-1)^j\beta_j}{x^j} \qquad (x \geq 1), \tag{3.10}$$

which develops the double inequality (3.3) to produce a general result. As an application of (3.10), Chen and Paris [10, Theorem 3.1] have given a generalized Carleman-type inequality.

In 2001 Yang [43] conjectured, then Yang [44], Gylletberg and Yan [16], Chen [5], Lü et al. [27], and Hu and Mortici [20] proved that if the following equality holds:

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right) \tag{3.11}$$

for $x > 0$, then $b_k > 0$ for $k \in \mathbb{N}$. In fact, Yang [44], Gylletberg and Yan [16], and Chen [5] presented the following recurrence relation for determining the coefficients b_k in (3.11):

$$b_1 = \frac{1}{2}, \quad b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2}-\sum_{j=1}^{n}\frac{b_j}{n+2-j}\right) \qquad (n \geq 1), \tag{3.12}$$

and then proved $b_k > 0$ for $k \in \mathbb{N}$; see also Lü et al. [27]. Hu and Mortici [20] used an argument of Alzer and Berg [2] to derive an integral representation for b_k, and then obtained some new properties of b_k, including $b_k > 0$ for $k \in \mathbb{N}$. We remark that the recurrence relation of the coefficients b_k given in [19, Lemma 2.2] is not correct.

Remark 3.1. *We give here an explicit formula for determining the coefficients* b_k *in* (3.11)*:*

$$b_j = -\sum_{(k_1,\,k_2,\,\dots,\,k_j)\in\mathcal{A}_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{1}{1\cdot 2}\right)^{k_1}\left(\frac{1}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{1}{j(j+1)}\right)^{k_j}, \tag{3.13}$$

where the $\mathcal{A}_j$ $(j\in\mathbb{N})$ *are given in* (2.1)*.*

Noting that $b_k > 0$ for $k\in\mathbb{N}$ in (3.11), it follows from (3.11) that

$$\left(1+\frac{1}{x}\right)^x < e\left(1-\sum_{k=1}^{m}\frac{b_k}{(1+x)^k}\right) \tag{3.14}$$

for $x > 0$ and $m \in \mathbb{N}$. As an application of (3.14), inequalities (3.2) and (3.1) were strengthened by Yang [44, Corollaries 2 and 3].

In the final part of his paper, Yang [43] remarked that in order to obtain better results, the right-hand side of (3.11) could be replaced by $e[1-\sum_{n=1}^{\infty}(d_n/(x+\varepsilon)^n)]$, where $\varepsilon\in(0,1]$ and $d_n = d_n(\varepsilon)$, but information about the values of ε are not provided. In fact, Xie and Zhong [39] proved in 2000 that $x \geq 1$,

$$e\left(1-\frac{7}{14x+12}\right) < \left(1+\frac{1}{x}\right)^x < e\left(1-\frac{6}{12x+11}\right), \tag{3.15}$$

and then applied it to obtain an improvement of (3.2) as follows: if $0<\lambda_{n+1}\leq\lambda_n$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m$, $a_n\geq 0$ $(n\in\mathbb{N})$ and $0<\sum_{n=1}^{\infty}\lambda_n a_n<\infty$, then

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\left(1-\frac{\frac{1}{2}}{\Lambda_n/\lambda_n+\frac{11}{12}}\right)\lambda_n a_n. \tag{3.16}$$

Recently, Mortici and Hu [32] gave a formula for determining the coefficients d_k such that

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= e\left(1-\sum_{k=1}^{\infty}\frac{d_k}{(\frac{11}{12}+n)^k}\right)\\ &= e\left(1-\frac{\frac{1}{2}}{n+\frac{11}{12}}-\frac{\frac{5}{288}}{(n+\frac{11}{12})^3}-\frac{\frac{139}{17280}}{(n+\frac{11}{12})^4}-\frac{\frac{119}{23040}}{(n+\frac{11}{12})^5}-\cdots\right),\end{aligned} \tag{3.17}$$

which is better than (3.11), since by truncation after $k\geq 3$ terms of series (3.11), the last term is of order $n^{-(k-1)}$, while the last term of series (3.17) truncated after k terms is of order n^{-k}. For the same reason, the formula (3.17) is better than (3.6).

Let

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right) = e\left(1-\sum_{k=1}^{\infty}\frac{d_k}{(\frac{11}{12}+x)^k}\right),$$

$$\sigma_m(x)=\sum_{k=1}^{m}\frac{b_k}{(1+x)^k} \quad\text{and}\quad S_m(x)=\sum_{k=1}^{m}\frac{d_k}{(\frac{11}{12}+x)^k}.$$

Then Ren and Li [36] proved that (i) if $m\geq 6$ is even, we have $S_m(x)>\sigma_m(x)$ for all $x>0$ and (ii) if $m\geq 7$ is odd, we have $S_m(x)>\sigma_m(x)$ for all $x>1$. This provides an intuitive explanation for the main result in Mortici and Hu [32].

Recently, You et al. [45] provided continued fraction inequalities related to $(1+1/x)^x$, which can be used to refine the inequalities (3.1) and (3.2).

3.2 A new form of approximation for $(1+1/x)^x$

Using the Maple software, we find[1]

$$\left(1+\frac{1}{x}\right)^x \sim e\left(1-\frac{\frac{1}{2}}{x+\frac{11}{12}}-\frac{\frac{5}{288}}{(x+\frac{343}{450})^3}-\frac{\frac{41683}{15552000}}{(x+\frac{558100391}{787808700})^5}-\cdots\right) \tag{3.18}$$

as $x\to\infty$. This led us to pose the following problem: Find the constants λ_ℓ and μ_ℓ such that

$$\left(1+\frac{1}{x}\right)^x \sim e\left(1+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)$$

as $x\to\infty$. In this section we solve this problem. Thus, we would appear to obtain an odd-type asymptotic expansion for $(1+1/x)^x$. From a computational viewpoint, (3.18) is an improvement on the formulas (3.6), (3.11) and (3.17).

Theorem 3.1. *As $x\to\infty$, we have*

$$\left(1+\frac{1}{x}\right)^x \sim e\left(1+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right), \tag{3.19}$$

where the constants λ_ℓ and μ_ℓ are given by the pair of recurrence relations

$$\lambda_\ell = a_{2\ell-1}-\sum_{k=1}^{\ell-1}\lambda_k\mu_k^{2\ell-2k}\binom{2\ell-2}{2\ell-2k} \qquad (\ell\geq 2) \tag{3.20}$$

and

$$\mu_\ell = -\frac{1}{(2\ell-1)\lambda_\ell}\left\{a_{2\ell}+\sum_{k=1}^{\ell-1}\lambda_k\mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}\right\} \qquad (\ell\geq 2), \tag{3.21}$$

with $\lambda_1=-\frac{1}{2}$ and $\mu_1=\frac{11}{12}$. Here a_j are given in (3.7).

Proof. We first express (3.19) in the form

$$e^{-1}\left(1+\frac{1}{x}\right)^x-1\sim\sum_{j=1}^{\infty}\frac{\lambda_j}{x^{2j-1}}\left(1+\frac{\mu_j}{x}\right)^{-2j+1}.$$

[1]Using the Maple software, formula (3.18) is given in the appendix.

Direct computation yields

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{\lambda_j}{x^{2j-1}}\left(1+\frac{\mu_j}{x}\right)^{-2j+1} &= \sum_{j=1}^{\infty}\frac{\lambda_j}{x^{2j-1}}\sum_{k=0}^{\infty}\binom{-2j+1}{k}\frac{\mu_j^k}{x^k}\\ &= \sum_{j=1}^{\infty}\frac{\lambda_j}{x^{2j-1}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j-2}{k}\frac{\mu_j^k}{x^k}\\ &= \sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\lambda_{k+1}\mu_{k+1}^{j-k-1}(-1)^{j-k-1}\binom{j+k-1}{j-k-1}\frac{1}{x^{j+k}}\\ &= \sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\lfloor\frac{j+2}{2}\rfloor}\lambda_k\mu_k^{j-2k+1}(-1)^{j-1}\binom{j-1}{j-2k+1}\right\}\frac{1}{x^j}.\end{aligned}$$

We then obtain

$$e^{-1}\left(1+\frac{1}{x}\right)^x-1\sim\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\lfloor\frac{j+2}{2}\rfloor}\lambda_k\mu_k^{j-2k+1}(-1)^{j-1}\binom{j-1}{j-2k+1}\right\}\frac{1}{x^j}. \tag{3.22}$$

On the other hand, it follows from (3.7) that

$$e^{-1}\left(1+\frac{1}{x}\right)^x-1=\sum_{j=1}^{\infty}\frac{a_j}{x^j}, \tag{3.23}$$

where a_j are given in (3.8). Equating coefficients of the term x^{-j} on the right-hand sides of (3.22) and (3.23), we obtain

$$a_j=\sum_{k=1}^{\lfloor\frac{j+2}{2}\rfloor}\lambda_k\mu_k^{j-2k+1}(-1)^{j-1}\binom{j-1}{j-2k+1}\qquad(j\in\mathbb{N}). \tag{3.24}$$

Setting $j=2\ell-1$ and $j=2\ell$ in (3.24), respectively, we find

$$a_{2\ell-1}=\sum_{k=1}^{\ell}\lambda_k\mu_k^{2\ell-2k}\binom{2\ell-2}{2\ell-2k} \tag{3.25}$$

and

$$\begin{aligned}a_{2\ell} &= -\sum_{k=1}^{\ell+1}\lambda_k\mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}\\ &= -\sum_{k=1}^{\ell}\lambda_k\mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}-\lambda_{\ell+1}\mu_{\ell+1}^{-1}\binom{2\ell-1}{-1}\\ &= -\sum_{k=1}^{\ell}\lambda_k\mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}.\end{aligned} \tag{3.26}$$

From (3.25) and (3.26) we obtain for $\ell = 1$,

$$\lambda_1 = a_1 = -\frac{1}{2} \quad \text{and} \quad \mu_1 = -\frac{a_2}{\lambda_1} = \frac{11}{12},$$

and for $\ell \geq 2$ we have

$$a_{2\ell-1} = \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \lambda_\ell$$

and

$$a_{2\ell} = -\sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - (2\ell-1)\lambda_\ell \mu_\ell.$$

We then obtain the recurrence relations (3.20) and (3.21). The proof is complete. □

We give explicit numerical values of the first few constants λ_ℓ and μ_ℓ by using the formulas (3.20) and (3.21). This demonstrates the ease with which the constants λ_ℓ and μ_ℓ in (3.19) can be determined.

$$\lambda_1 = -\frac{1}{2}, \quad \mu_1 = \frac{11}{12},$$

$$\lambda_2 = a_3 - \lambda_1 \mu_1^2 = -\frac{7}{16} - \left(-\frac{1}{2}\right) \cdot \left(\frac{11}{12}\right)^2 = -\frac{5}{288},$$

$$\mu_2 = -\frac{a_4 + \lambda_1 \mu_1^3}{3\lambda_2} = -\frac{\frac{2447}{5760} + \left(-\frac{1}{2}\right) \cdot \left(\frac{11}{12}\right)^3}{3 \cdot \left(-\frac{5}{288}\right)} = \frac{343}{450},$$

$$\lambda_3 = a_5 - \lambda_1 \mu_1^4 - 6\lambda_2 \mu_2^2 = -\frac{959}{2304} - \left(-\frac{1}{2}\right) \cdot \left(\frac{11}{12}\right)^4 - 6 \cdot \left(-\frac{5}{288}\right) \cdot \left(\frac{343}{450}\right)^2 = -\frac{41683}{15552000},$$

$$\begin{aligned} \mu_3 &= -\frac{a_6 + \lambda_1 \mu_1^5 + 10\lambda_2 \mu_2^3}{5\lambda_3} \\ &= -\frac{\frac{238043}{580608} + \left(-\frac{1}{2}\right) \cdot \left(\frac{11}{12}\right)^5 + 10 \cdot \left(-\frac{5}{288}\right) \cdot \left(\frac{343}{450}\right)^3}{5 \cdot \left(-\frac{41683}{15552000}\right)} = \frac{558100391}{787808700}. \end{aligned}$$

We note that the values of λ_ℓ and μ_ℓ (for $\ell = 1, 2, 3$) above are equal to the constants appearing in (3.18).

Remark 3.2. *By using the Maple software, we can show that for $x > 0$,*

$$\left(1 + \frac{1}{x}\right)^x < e \left(1 - \frac{\frac{1}{2}}{x + \frac{11}{12}} - \frac{\frac{5}{288}}{(x + \frac{343}{450})^3} - \frac{\frac{41683}{15552000}}{(x + \frac{558100391}{787808700})^5}\right). \tag{3.27}$$

We omit the proof.

By virtue of the proof given in [42] and the inequality (3.27)*, we have the Carleman-type inequality*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < \sum_{n=1}^{\infty} \left(1 + \frac{1}{\Lambda_n/\lambda_n}\right)^{\Lambda_n/\lambda_n} \lambda_n a_n$$
$$< e \sum_{n=1}^{\infty} \left(1 - \frac{\frac{1}{2}}{(\Lambda_n/\lambda_n) + \frac{11}{12}} - \frac{\frac{5}{288}}{\left((\Lambda_n/\lambda_n) + \frac{343}{450}\right)^3} - \frac{\frac{41683}{15552000}}{\left((\Lambda_n/\lambda_n) + \frac{558100391}{787808700}\right)^5}\right) \lambda_n a_n, \quad (3.28)$$

which is an improvement on the inequality (3.16)*.*

Finally, we propose the following conjecture.

Conjecture 3.1. For all $\ell \in \mathbb{N}$, we have

$$\lambda_\ell < 0 \quad \text{and} \quad \mu_\ell > 0. \quad (3.29)$$

Further, we have the inequality

$$\left(1 + \frac{1}{x}\right)^x < e \left(1 + \sum_{\ell=1}^{m} \frac{\lambda_\ell}{(x + \mu_\ell)^{2\ell-1}}\right) \quad (3.30)$$

for $x > 0$ and $m \in \mathbb{N}$.

Appendix: A derivation of formula (3.18)

Define the function $F(x)$ by

$$F(x) = \left(1 + \frac{1}{x}\right)^x - e \left(1 + \frac{\lambda_1}{x + \mu_1}\right).$$

We are interested in finding the values of the parameters λ_1 and μ_1 such that $F(x)$ converges as fast as possible to zero, as $x \to \infty$. This provides the best approximation of the form:

$$\left(1 + \frac{1}{x}\right)^x \approx e \left(1 + \frac{\lambda_1}{x + \mu_1}\right).$$

Using the Maple software, we find, as $x \to \infty$,

$$F(x) = -\frac{e(1 + 2\lambda_1)}{2x} + \frac{e(24\lambda_1\mu_1 + 11)}{24x^2} - \frac{e(16\lambda_1\mu_1^2 + 7)}{16x^3} + O\left(\frac{1}{x^4}\right).$$

The two parameters λ_1 and μ_1, which produce the fastest convergence of the function $F(x)$, are given by

$$\begin{cases} 1 + 2\lambda_1 = 0 \\ 24\lambda_1\mu_1 + 11 = 0, \end{cases}$$

namely, if

$$\lambda_1 = -\frac{1}{2}, \quad \mu_1 = \frac{11}{12}.$$

We then obtain, as $x \to \infty$,

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \frac{\frac{1}{2}}{x+\frac{11}{12}} + O\left(\frac{1}{x^3}\right)\right). \tag{3.31}$$

In view of (3.31), we define the function $G(x)$ by

$$G(x) = \left(1+\frac{1}{x}\right)^x - e\left(1 - \frac{\frac{1}{2}}{x+\frac{11}{12}} + \frac{\lambda_2}{(x+\mu_2)^3}\right).$$

Using the Maple software, we find, as $x \to \infty$,

$$G(x) = -\frac{e(5+288\lambda_2)}{288x^3} + \frac{e(343+25920\lambda_2\mu_2)}{8640x^4} - \frac{e(2621+248832\lambda_2\mu_2^2)}{41472x^5} + O\left(\frac{1}{x^6}\right).$$

For $\lambda_2 = -\frac{5}{288}$ and $\mu_2 = \frac{343}{450}$, we obtain, as $x \to \infty$,

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \frac{\frac{1}{2}}{x+\frac{11}{12}} - \frac{\frac{5}{288}}{(x+\frac{343}{450})^3} + O\left(\frac{1}{x^5}\right)\right). \tag{3.32}$$

In view of (3.32), we define the function $H(x)$ by

$$H(x) = \left(1+\frac{1}{x}\right)^x - e\left(1 - \frac{\frac{1}{2}}{x+\frac{11}{12}} - \frac{\frac{5}{288}}{(x+\frac{343}{450})^3} + \frac{\lambda_3}{(x+\mu_3)^5}\right).$$

Using the Maple software, we find, as $x \to \infty$,

$$\begin{aligned} H(x) = &-\frac{e(41683+15552000\lambda_3)}{15552000x^5} + \frac{e(558100391+293932800000\lambda_3\mu_3)}{58786560000x^6} \\ &- \frac{e(52111420409+37791360000000\lambda_3\mu_3^2)}{2519424000000x^7} + O\left(\frac{1}{x^8}\right). \end{aligned}$$

For $\lambda_3 = -\frac{41683}{15552000}$ and $\mu_3 = \frac{558100391}{787808700}$, we obtain, as $x \to \infty$,

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \frac{\frac{1}{2}}{x+\frac{11}{12}} - \frac{\frac{5}{288}}{(x+\frac{343}{450})^3} - \frac{\frac{41683}{15552000}}{(x+\frac{558100391}{787808700})^5} + O\left(\frac{1}{x^7}\right)\right).$$

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