Evaluations of some terminating hypergeometric $2F_1(2)$ series with applications

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Abstract: Explicit expressions for the hypergeometric series $2F_1(-n, a; 2a \pm j; 2)$ and $2F_1(-n, a; -2n \pm j; 2)$ for positive integer $n$ and arbitrary integer $j$ are obtained with the help of generalizations of Kummer's second and third summation theorems obtained earlier by Rakha and Rathie. Results for $|j| \leq 5$ derived previously using different methods are also obtained as special cases. Two applications are considered, where the first summation formula is applied to a terminating $3F_2(2)$ series and the confluent hypergeometric function $1F_1(x)$.

Key words: Terminating hypergeometric series, generalized Kummer’s second and third summation theorems

1. Introduction

In a problem arising in a model of a biological problem, Samoletov [14] obtained by means of a mathematical induction argument the following sum containing factorials:

$$\sum_{k=0}^{n} \frac{(-1)^k (2k + 1)!}{(n-k)! k!(k+1)!} = \frac{(-1)^n}{\sqrt{n!(n+1)!}} \left( \sqrt{n+1} \frac{(n-1)!!}{n!!} \right)^{(-1)^n},$$

where throughout $n$ denotes a positive integer and, as usual,

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \quad (2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n n!}.$$

Samoletov [14] also expressed the above sum in the equivalent hypergeometric form:

$$2F_1\left[ -\frac{n}{2}, \frac{3}{2}; 2 \right] = \begin{cases} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}+1\right)}, & (n \text{ even}) \\ -\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}+\frac{3}{2}\right)}, & (n \text{ odd}) \end{cases}$$

Subsequently, Srivastava [17] pointed out that this result could be easily derived from a known hypergeometric summation formula [11, Vol. 2, p. 493] for $2F_1(-n, a; 2a - 1; 2)$ with $a = \frac{3}{2}$, which is a contiguous result to the

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The well-known summation theorem was originally found by Kummer. This involved expressing the series for $j$ and $\frac{a}{2} \pm j$ for arbitrary integer $j$. We shall employ the following generalizations of Kummer's second and third summation theorems given in [13] (we correct a misprint in Theorem 6 of this reference). These are respectively

$$2F_1 \left[ \frac{\alpha, \beta}{\frac{1}{2}(\alpha + \beta + j + 1)} : \frac{1}{2} \right] = \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \beta + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \alpha - \frac{1}{2} \beta + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \beta + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \alpha - \frac{1}{2} \beta + \frac{1}{2} \right)} \times \sum_{r=0}^{j} (\mp 1)^r \binom{j}{r} \left( \frac{\beta}{\alpha + \frac{1}{2}r} \right)^{j-r}$$

(3)

and

$$2F_1 \left[ \frac{\alpha, 1 - \alpha \pm j}{\gamma} : \frac{1}{2} \right] = \frac{2^{\pm j} \Gamma \left( \frac{1}{2} \gamma \right) \Gamma \left( \frac{1}{2} \gamma + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \gamma + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \gamma - \frac{1}{2} \alpha + \frac{1}{2} \right) \Gamma \left( \alpha + \epsilon_j \right)} \times \sum_{r=0}^{j} (\mp 1)^r \binom{j}{r} \left( \frac{1}{2} \gamma - \frac{1}{2} \alpha \right)^{r/2}$$

(4)

for $j = 0, 1, 2, \ldots$, where $\epsilon_j = 0$ (resp. $j$), $\delta_j = j$ (resp. 0) when the upper (resp. lower) signs are taken and $(a)_k = \Gamma(a + k)/\Gamma(a)$ is the Pochhammer symbol defined for arbitrary index $k$. The formulas (3) (for $j > 0$) and (4) (separately for $j > 0$ and $j < 0$) are given in a different form in [1, p. 582, (130)–(132)]. When $j = 0$, the summations (3) and (4) reduce to the well-known second and third summation theorems due to Kummer [16, p. 243]:

$$2F_1 \left[ \frac{\alpha, \beta}{\frac{1}{2}(\alpha + \beta + 1)} : \frac{1}{2} \right] = \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \beta + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \alpha - \frac{1}{2} \beta + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \beta + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \alpha - \frac{1}{2} \beta + \frac{1}{2} \right)}$$

and

$$2F_1 \left[ \frac{\alpha, 1 - \alpha \pm j}{\gamma} : \frac{1}{2} \right] = \frac{\Gamma \left( \frac{1}{2} \gamma \right) \Gamma \left( \frac{1}{2} \gamma + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \gamma + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \gamma - \frac{1}{2} \alpha + \frac{1}{2} \right) \Gamma \left( \alpha \mp j \right)}$$

In addition, we shall make use of the transformation [10, (15.8.6)]

$$2F_1 \left[ \frac{-n, \beta}{\gamma} : \frac{1}{2} \right] = \frac{(-2)^n \beta_n}{\gamma_n} 2F_1 \left[ \frac{-n, 1 - \gamma - n}{1 - \beta - n} : \frac{1}{2} \right].$$

(5)

Expressions for the series in (2) for arbitrary integer $j$ were recently obtained by Chu [3] using a different approach. This involved expressing the series for $j \neq 0$ as finite sums of $2F_1(2)$ series in (2) with $j = 0$. The

*In [16, p. 243], this summation formula is referred to as Bailey’s theorem. However, it has been pointed out in [2] that this theorem was originally found by Kummer.
cases with $|j| \leq 5$ were also given by Kim and Rathie [6] and Kim et al. [5]. An application of the first series in (2) for $j = 0, 1, \ldots, 5$ was discussed in [7]. In Sections 4 and 5, we give two additional applications, the first to the evaluation of a terminating $_3F_2(2)$ series and the second to the confluent hypergeometric function $_1F_1(x)$.

2. Statement of the results

Our principal results are stated in the following two theorems.

Theorem 1 Let $n$ be a positive integer, let $a$ be a complex parameter, and define $j_0 = \lfloor \frac{1}{2} j \rfloor$. Then we have

$$
_2F_1 \left[ \begin{array}{c} -2n, a \\ 2a \pm j \end{array} ; 2 \right] = \frac{2^{2n} \binom{1}{2} n}{(2a \pm j)^{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a + \delta_j)_{n-r} \quad (8)
$$

and

$$
_2F_1 \left[ \begin{array}{c} -2n - 1, a \\ 2a \pm j \end{array} ; 2 \right] = \frac{\pm 2^{2n} \binom{1}{2} n}{(2a \pm j)^{2n+1}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r+1} (-n)_r (a + \delta_j)_{n-r} \quad (9)
$$

for $j = 0, 1, 2, \ldots$, where $\delta_j = j$ (resp. 0) when the upper (resp. lower) signs are taken.

Proof From the result (5) we have

$$
_2F_1 \left[ \begin{array}{c} -n, a \\ 2a \pm j \end{array} ; 2 \right] = \frac{(-2)^n a_n}{(2a \pm j)^n} _2F_1 \left[ \begin{array}{c} -n, 1 - 2a \mp j - n \\ 1 - a - n \end{array} ; 1 \right] .
$$

The hypergeometric function on the right-hand side can be summed by the generalized second Kummer summation theorem (3). Proceeding first with the function whose denominator parameter is $2a + j$, we put $\alpha = 1 - 2a - j - n$ and $\beta = -n$ in (3) to find, after some straightforward algebra,

$$
_2F_1 \left[ \begin{array}{c} -n, a \\ 2a + j \end{array} ; 2 \right] = \frac{(-2)^n a_n \sqrt{\pi} \Gamma(1 - a - n)}{(2a + j) \Gamma(-\frac{1}{2} n + \frac{1}{2}) \Gamma(1 - a)} \sum_{r=0}^{j_0} (-1)^r \binom{j}{r} \frac{(-\frac{1}{2} n)_{r/2}}{(1 - a - j)_{(r-n)/2}} .
$$

Now replace $n$ by $2n$ and use the reflection formula for the gamma function to yield

$$
_2F_1 \left[ \begin{array}{c} -2n, a \\ 2a + j \end{array} ; 2 \right] = \frac{(-1)^{n} 2^{2n} \binom{1}{2} n}{(2a + j)^{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{r} \frac{(-n)_{r/2}}{(1 - a - j)_{r/2 - n}} .
$$

Since $(-n)_{r/2}$ vanishes for odd $r$ and positive integer $n$, we finally obtain

$$
_2F_1 \left[ \begin{array}{c} -2n, a \\ 2a + j \end{array} ; 2 \right] = \frac{2^{2n} \binom{1}{2} n}{(2a + j)^{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (n)_{r}(a + j)_{n-r} ,
$$

where $j_0 = \lfloor \frac{1}{2} j \rfloor$ and we have used the fact that $(1 - a - j)_{r-n} = (-1)^{n-r}/(a + j)_{n-r}$.
If we replace $n$ by $2n + 1$ in (8), we find
\[
\ {_2F_1} \left[ \begin{array}{c} -2n - 1, a \\ 2a + j \\end{array} ; 2 \right] = \frac{2^{2n+1}}{(2a + j)_{2n+1}} \frac{\sqrt{\pi}}{\Gamma(-n - \frac{1}{2})} \sum_{r=0}^{j} (-1)^{r} \binom{j}{r} \frac{(-n)_{(r-1)/2}}{(1-a-j)_{(r-1)/2-n}},
\]
where we have used
\[
\frac{(-n - \frac{1}{2})_{r/2}}{\Gamma(-n - \frac{1}{2})} = \frac{(-n)_{(r-1)/2}}{\Gamma(-n - \frac{1}{2})}.
\]
Since $(-n)_{(r-1)/2}$ vanishes for even $r$ and positive integer $n$, we find
\[
\ _2F_1 \left[ \begin{array}{c} -2n - 1, a \\ 2a + j \\end{array} ; 2 \right] = \frac{2^{2n}(\frac{3}{2})_{n}}{(2a + j)_{2n+1}} \sum_{r=0}^{j} (-1)^{r} \binom{j}{2r+1} (-n)_{r}(a+j)_{n-r}.
\]
(10)
Proceeding in a similar manner for the function whose denominator parameter is $2a - j$, we have upon letting $\alpha = 1 - a + j - n$ and $\beta = -n$ in (3)
\[
\ _2F_1 \left[ \begin{array}{c} -n, a \\ 2a - j \\end{array} ; 2 \right] = \frac{2^{n}}{(2a - j)_{n}} \frac{\sqrt{\pi}}{\Gamma(-\frac{n}{2} + \frac{1}{2})} \sum_{r=0}^{j} \binom{j}{r} \frac{(-\frac{1}{2} n)_{r/2}}{(1-a)_{(r-n)/2}}.
\]
(11)
Then we have
\[
\ _2F_1 \left[ \begin{array}{c} -2n - 1, a \\ 2a - j \\end{array} ; 2 \right] = \frac{2^{2n}(\frac{3}{2})_{n}}{(2a - j)_{2n}} \sum_{r=0}^{j} (-1)^{r} \binom{j}{2r} (-n)_{r}(a+j)_{n-r}
\]
(12)
and
\[
\ _2F_1 \left[ \begin{array}{c} -2n - 1, a \\ 2a - j \\end{array} ; 2 \right] = -\frac{2^{2n}(\frac{3}{2})_{n}}{(2a - j)_{2n+1}} \sum_{r=0}^{j} (-1)^{r} \binom{j}{2r+1} (-n)_{r}(a)_{n-r}.
\]
(13)
The summations in (9), (10) and (12), (13) then correspond to the results stated in the theorem.

We remark that in (7) the upper limit of summation can be replaced by $\lceil \frac{1}{2} j \rceil - 1$ when $j$ is even. Also, since $(-n)_r$ vanishes when $r > n$, it is possible to replace the upper summation limit in both (6) and (7) by $n$ whenever $n > \lceil \frac{1}{2} j \rceil$.

**Theorem 2** Let $n$ be a positive integer and $a$ be a complex parameter. Then we have
\[
\ _2F_1 \left[ \begin{array}{c} -n, a \\ -2n + j \\end{array} ; 2 \right] = \frac{2^{2n-j}(n-j)!}{(2n-j)!} \sum_{r=0}^{j} \binom{j}{r} (\frac{1}{2} a + \frac{1}{2} - \frac{1}{2} r)_n
\]
(14)
provided that $j$ does not lie in the interval $[n+1, 2n]$ (where the hypergeometric function on the left-hand side of (14) is, in general, not defined), and
\[
\ _2F_1 \left[ \begin{array}{c} -n, a \\ -2n - j \\end{array} ; 2 \right] = \frac{2^{2n+j}n!}{(2n + j)!} \sum_{r=0}^{j} (-1)^{r} \binom{j}{r} (\frac{1}{2} a + \frac{1}{2} - \frac{1}{2} r)_{n+j}
\]
(15)
for $j = 0, 1, 2, \ldots$. When $j \geq 2n + 1$ in (14), the ratio of factorials $(n-j)!/(2n-j)!$ can be replaced by $(-1)^n(j-2n-1)!/(j-n-1)!$.
Proof From the result (5) we have

\[ 2F_1 \left[ \begin{array}{l} -n, a \\ -2n + j \end{array} : 2 \right] = \frac{2^n(a)_n(n \pm j)!}{(2n \pm j)!} 2F_1 \left[ \begin{array}{l} -n, 1 + n \pm j \\ 1 - a - n \end{array} : 2 \right] \]

when the parameters are such that the hypergeometric functions make sense. The hypergeometric function on the right-hand side can be summed by the generalized third Kummer theorem (4), where we put \( \alpha = -n \) and \( \gamma = 1 - a - n \). Then we obtain

\[ 2F_1 \left[ \begin{array}{l} -n, a \\ -2n + j \end{array} ; 2 \right] = \frac{2^{n-j}(a)_n(n-j)!}{(2n-j)!} \frac{\Gamma\left(\frac{1}{2}\gamma\right)\Gamma\left(\frac{1}{2}\gamma + j\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}a\right)\Gamma\left(1-\frac{1}{2}a\right)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}a + \frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}a + \frac{1}{2}r - n\right)} \]

Similarly, we find

\[ 2F_1 \left[ \begin{array}{l} -n, a \\ -2n - j \end{array} ; 2 \right] = \frac{(-1)^j 2^{n+j}(a)_n(n+j)!}{(2n+j)!} \frac{\Gamma\left(\frac{1}{2}\gamma\right)\Gamma\left(\frac{1}{2}\gamma + 1\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}a\right)\Gamma\left(1-\frac{1}{2}a\right)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}a + \frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}a + \frac{1}{2}r - n-j\right)} \]

which establishes the theorem. \( \square \)

The sums on the right-hand sides of (14) and (15) can be written in an alternative form involving just two Pochhammer symbols containing the index \( n \) by making use of the result

\[ (\alpha - r)_n = \frac{(\alpha)_n(1-\alpha)_r}{(1-\alpha - n)_r} \]

for positive integers \( r \) and \( n \). Then we find, with \( j_0 = \lfloor \frac{1}{2} j \rfloor \),

\[ 2F_1 \left[ \begin{array}{l} -n, a \\ -2n + j \end{array} ; 2 \right] = \frac{2^{n-j}(n-j)!}{(2n-j)!} \left\{ \left( \frac{1}{2}a + \frac{1}{2} \right)_n \sum_{r=0}^{j_0} \binom{j}{2r} A_r(n,0) + \left( \frac{1}{2}a \right)_n \sum_{r=0}^{j_0} \binom{j}{2r+1} B_r(n,0) \right\} \]  

(16)
\[\begin{align*}
\text{and} & \quad {_{2}F_{1}}\left[ \begin{array}{c}
-n, a \\
-2n - j
\end{array} ; 2 \right] \\
& \quad = \frac{2^{2n+j}n!}{(2n+j)!} \left\{ \left( \frac{1}{2} a + \frac{1}{2} \right)_{n+j} \sum_{r=0}^{n} \left( \begin{array}{c}
2r \\
2r + 1
\end{array} \right) A_r(n, j) - \left( \frac{1}{2} a \right)_{n+j} \sum_{r=0}^{n} \left( \begin{array}{c}
2r + 1 \\
2r
\end{array} \right) B_r(n, j) \right\}, \quad (17)
\end{align*}\]

where

\[A_r(n, j) := \left( \frac{1}{2} - \frac{1}{2} a \right) \left( \frac{1}{2} - a - n - j \right)_r, \quad B_r(n, j) := \left( 1 - \frac{1}{2} a \right) \left( 1 - \frac{1}{2} a - n - j \right)_r.\]

Again, when \( j \) is even, the upper summation limit in the second sums in (16) and (17) can be replaced by \( j_0 - 1 \), if so desired.

3. Special cases

If we set \( 0 \leq j \leq 5 \) in (6) we obtain the following summations:

\[\begin{align*}
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a + \frac{1}{2})_n} = \quad {_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a + 1
\end{array} ; 2 \right], \quad (18) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a + 2
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a + \frac{3}{2})_n} \left( 1 + \frac{2n}{a + 1} \right), \quad (19) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a + 3
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a + \frac{5}{2})_n} \left( 1 + \frac{4n}{a + 2} \right), \quad (20) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a + 4
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a + \frac{7}{2})_n} \left( 1 + \frac{8n}{a + 2} + \frac{8n(n-1)}{(a + 2)(a + 3)} \right), \quad (21) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a + 5
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a + \frac{9}{2})_n} \left( 1 + \frac{12n}{a + 3} + \frac{16n(n-1)}{(a + 3)(a + 4)} \right) \quad (22)
\end{align*}\]

and

\[\begin{align*}
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a - 1
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a - \frac{1}{2})_n}, \quad (23) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a - 2
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a - \frac{3}{2})_n} \left( 1 + \frac{2n}{a - 1} \right), \quad (24) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a - 3
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a - \frac{5}{2})_n} \left( 1 + \frac{4n}{a - 1} \right), \quad (25) \\
{_{2}F_{1}}\left[ \begin{array}{c}
-2n, a \\
2a - 4
\end{array} ; 2 \right] & = \frac{(\frac{1}{2})_n}{(a - \frac{7}{2})_n} \left( 1 + \frac{8n}{a - 2} + \frac{8n(n-1)}{(a - 1)(a - 2)} \right), \quad (26)
\end{align*}\]
2F\(_1\left[\frac{-2n, a}{2a - 5}; 2\right]\) = \frac{\left(\frac{1}{2}\right)_n}{(a - \frac{5}{2})_n} \left(1 + \frac{12n}{a - 2} + \frac{16n(n - 1)}{(a - 1)(a - 2)}\right). \tag{27}

Similarly, if we set 0 \leq j \leq 5 in (7) we obtain the following summations:

2F\(_1\left[\frac{-2n - 1, a}{2a}; 2\right]\) = 0, \tag{28}

2F\(_1\left[\frac{-2n - 1, a}{2a + 1}; 2\right]\) = \frac{\left(\frac{3}{2}\right)_n}{(2a + 1)(a + \frac{3}{2})_n}, \tag{29}

2F\(_1\left[\frac{-2n - 1, a}{2a + 2}; 2\right]\) = \frac{2\left(\frac{3}{2}\right)_n}{(2a + 2)(a + \frac{3}{2})_n}, \tag{30}

2F\(_1\left[\frac{-2n - 1, a}{2a + 3}; 2\right]\) = \frac{\left(\frac{3}{2}\right)_n}{(2a + 3)(a + \frac{3}{2})_n} \left(3 + \frac{4n}{a + 2}\right), \tag{31}

2F\(_1\left[\frac{-2n - 1, a}{2a + 4}; 2\right]\) = \frac{\left(\frac{3}{2}\right)_n}{(2a + 4)(a + \frac{3}{2})_n} \left(4 + \frac{8n}{a + 3}\right), \tag{32}

2F\(_1\left[\frac{-2n - 1, a}{2a + 5}; 2\right]\) = \frac{\left(\frac{3}{2}\right)_n}{(2a + 5)(a + \frac{3}{2})_n} \left(5 + \frac{20n}{a + 3} + \frac{16n(n - 1)}{(a + 3)(a + 4)}\right), \tag{33}

and

2F\(_1\left[\frac{-2n - 1, a}{2a - 1}; 2\right]\) = -\frac{\left(\frac{3}{2}\right)_n}{(2a - 1)(a + \frac{3}{2})_n}, \tag{34}

2F\(_1\left[\frac{-2n - 1, a}{2a - 2}; 2\right]\) = -\frac{2\left(\frac{3}{2}\right)_n}{(2a - 2)(a - \frac{1}{2})_n}, \tag{35}

2F\(_1\left[\frac{-2n - 1, a}{2a - 3}; 2\right]\) = -\frac{\left(\frac{3}{2}\right)_n}{(2a - 3)(a - \frac{1}{2})_n} \left(3 + \frac{4n}{a - 1}\right), \tag{36}

2F\(_1\left[\frac{-2n - 1, a}{2a - 4}; 2\right]\) = -\frac{\left(\frac{3}{2}\right)_n}{(2a - 4)(a - \frac{3}{2})_n} \left(4 + \frac{8n}{a - 1}\right), \tag{37}

2F\(_1\left[\frac{-2n - 1, a}{2a - 5}; 2\right]\) = -\frac{\left(\frac{3}{2}\right)_n}{(2a - 5)(a - \frac{3}{2})_n} \left(5 + \frac{20n}{a - 2} + \frac{16n(n - 1)}{(a - 1)(a - 2)}\right), \tag{38}

Finally, from (16) and (17) we obtain:

2F\(_1\left[\frac{-n, a}{-2n}; 2\right]\) = \frac{2^{2n} n!}{(2n)!} \left(\frac{1}{2} a + \frac{1}{2}\right)_n = \frac{\left(\frac{1}{2} a + \frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n}, \tag{39}
From the results in Section 2 we shall establish a summation formula for the terminating \( 3 \binom{m}{3} \) series

\[
\binom{2n-1}{(n-1)!(2n+1)!} \left\{ \left( \frac{1}{2} a + \frac{1}{2} x \right)^n + \left( \frac{1}{2} a - \frac{1}{2} x \right)^n \right\}
\]

(40)

(\( \binom{m}{3} \)) have been obtained previously by different means in [4, 12]. When \( m = p = 0 \) the above series reduces to that in (1).

We employ a general result proved in [9, Lemma 4] expressing an \( \binom{m}{r} \) hypergeometric function, with \( r \) pairs of numeratorial and denominatorial parameters differing by positive integers, as a finite sum of \( \binom{m}{3} \) functions. In the particular case \( r = 1 \), we obtain for positive integer \( m \)

\[
\binom{m}{r} \sum_{j=0}^{m} A_j \binom{a}{j} \binom{b}{j} \binom{d}{j} \binom{c}{j} \binom{x}{j} \binom{d+j}{c+j} \binom{a+j}{d+j} \binom{x+j}{c+j}.
\]

(44)

The coefficients \( A_j \) are defined by

\[
A_j = \sum_{k=j}^{m} \sigma_{m-k} S_k^{(j)},
\]

where \( S_k^{(j)} \) is the Stirling number of the second kind and the \( \sigma_j \) (\( 0 \leq j \leq m \)) are generated by the relation

\[
(d + x)_m = \sum_{j=0}^{m} \sigma_{m-j} x^j.
\]

It was shown in [8] that

\[
A_j = \frac{(-1)^j (d+m)_j}{j!} \binom{a+m}{d} \binom{b+m}{c} \binom{c+m}{d} \binom{d+m}{a} \binom{a+j}{d+j} \binom{b+j}{c+j} \binom{c+j}{d+j} \binom{d+j}{a+j}.
\]

(45)
the second result following by application of Vandermonde’s theorem [16, p. 243].

In order to simplify the presentation of our summation theorem we introduce the coefficients \( C_r(n, j) \) given by

\[
C_r(n, j) = \frac{\Gamma(-\frac{1}{2} n + \frac{1}{2} j + \frac{1}{2} r)}{\Gamma(-\frac{1}{2} n + \frac{1}{2} j + \frac{1}{2})},
\]

which yield the even- and odd-order values

\[
C_{2r}(n, j) = \frac{(-\frac{1}{2} n + \frac{1}{2} j)_r}{\Gamma(-\frac{1}{2} n + \frac{1}{2} j + \frac{1}{2})}, \quad C_{2r+1}(n, j) = \frac{(-\frac{1}{2} n + \frac{1}{2} j + \frac{1}{2})_r}{\Gamma(-\frac{1}{2} n + \frac{1}{2} j + \frac{1}{2})}.
\]

We further introduce the finite sums for \( 0 \leq j \leq m \):

\[
S^{(1)}_j(n, p) = \sum_{r=0}^{n-j} \binom{p-j}{r} \frac{(-1)^r C_r(n, j)}{(1-a-p)(r-n+j)/2} \quad (p \geq j),
\]

\[
S^{(2)}_j(n, p) = \sum_{r=0}^{j-p} \binom{j-p}{r} \frac{C_r(n, j)}{(1-a-j)(r-n+j)/2} \quad (p \leq j).
\]

Then we obtain the following theorem.

**Theorem 3** Let \( m, n \) be positive integers and \( p \) be an arbitrary integer such that \( 2a+p \neq 0, -1, -2, \ldots, -n+1 \).

Then

\[
\begin{aligned}
3F_2 \left[ \begin{array}{c}
-n, a, \\
2a + p,
\end{array} \; \begin{array}{c}
d + m \\
d
\end{array} ; 2 \right] &= \left\{ \begin{array}{ll}
\frac{2^n \sqrt{\pi}}{(2a + p)_n} \sum_{j=0}^{m} \frac{(-1)^j (-m)_j (-n)_j (a)_j}{j! (d)_j} S^{(1)}_j(n, p) & (p \geq m > 0) \\
\frac{2^n \sqrt{\pi}}{(2a + p)_n} \left( \sum_{j=0}^{p} \frac{(-1)^j (-m)_j (-n)_j (a)_j}{j! (d)_j} S^{(1)}_j(n, p) + \sum_{j=p+1}^{m} \frac{(-1)^j (-m)_j (-n)_j (a)_j}{j! (d)_j} S^{(2)}_j(n, p) \right) & (0 \leq p < m) \\
\frac{2^n \sqrt{\pi}}{(2a + p)_n} \sum_{j=0}^{m} \frac{(-1)^j (-m)_j (-n)_j (a)_j}{j! (d)_j} S^{(2)}_j(n, p) & (p \leq 0).
\end{array} \right.
\end{aligned}
\]

**Proof** If we put \( b = -n, c = 2a + p \) and \( x = 2 \) in (44), combined with (45), we find

\[
3F_2 \left[ \begin{array}{c}
-n, a, \\
2a + p,
\end{array} \; \begin{array}{c}
d + m \\
d
\end{array} ; 2 \right] = \sum_{j=0}^{m} \frac{(-2)^j (-m)_j (-n)_j (a)_j}{j! (2a + p)_j (d)_j} 2F_1 \left[ \begin{array}{c}
-n + j, a + j \\
2a + p + j
\end{array} ; 2 \right].
\]

Substitution of (8) and (11) with \( n \to n - j, a \to a + j \) and the parameter in the denominator replaced by \( 2(a + j) + p - j \), followed by some straightforward algebra to distinguish between the cases with \( p - j \geq 0 \) and \( p - j < 0 \), leads to the results stated in the theorem. \( \square \)

We conclude this section by giving some specific examples of the summation in Theorem 3. Alternatively, these results may be obtained from (44), (45), and the \( 2F_1(2) \) summations listed in Section 3. For convenience in presentation we define

\[
\varphi_k := \frac{(a)_k}{(d)_k} \quad (k = 1, 2, \ldots).
\]
When \( m = p \), we have for \( 1 \leq m \leq 4 \) and positive integer \( n \)

\[
3F2 \left[ \frac{-2n, a, \; d + 1}{2a + 1, \; d} \right] = \frac{\left( \frac{1}{2} \right)_n}{(a + \frac{1}{2})_n}, \tag{46}
\]

\[
3F2 \left[ \frac{-2n - 1, a, \; d + 1}{2a + 1, \; d} \right] = \frac{\left( \frac{3}{2} \right)_n}{(2a + 1)(a + \frac{1}{2})_n} (1 - 2\psi_1), \tag{47}
\]

\[
3F2 \left[ \frac{-2n, a, \; d + 2}{2a + 2, \; d} \right] = \frac{\left( \frac{1}{2} \right)_n}{(a + \frac{1}{2})_n} \left( 1 + \frac{2n}{a + 1} [1 - 2\psi_1 + 2\psi_2] \right), \tag{48}
\]

\[
3F2 \left[ \frac{-2n - 1, a, \; d + 2}{2a + 2, \; d} \right] = \frac{\left( \frac{3}{2} \right)_n}{(a + 1)(a + \frac{1}{2})_n} \left( 1 - 2\psi_1 \right), \tag{49}
\]

\[
3F2 \left[ \frac{-2n, a, \; d + 3}{2a + 3, \; d} \right] = \frac{\left( \frac{1}{2} \right)_n}{(2a + 3)(a + \frac{1}{2})_n} \left( 1 + \frac{4n}{a + 2} (1 - 3\psi_1 + 3\psi_2) \right), \tag{50}
\]

\[
3F2 \left[ \frac{-2n - 1, a, \; d + 3}{2a + 3, \; d} \right] = \frac{\left( \frac{3}{2} \right)_n}{(2a + 3)(a + \frac{1}{2})_n} \left( 3 - 6\psi_1 + \frac{4n}{a + 2} (1 - 3\psi_1 + 3\psi_2 - 2\psi_3) \right), \tag{51}
\]

\[
3F2 \left[ \frac{-2n, a, \; d + 4}{2a + 4, \; d} \right]
= \frac{\left( \frac{1}{2} \right)_n}{(a + \frac{5}{2})_n} \left( 1 + \frac{8n}{a + 2} (1 - 3\psi_1 + 3\psi_2) + \frac{8n(n - 1)}{(a + 2)(a + 3)} (1 - 4\psi_1 + 6\psi_2 - 4\psi_3 + 2\psi_4) \right), \tag{52}
\]

and

\[
3F2 \left[ \frac{-2n - 1, a, \; d + 4}{2a + 4, \; d} \right] = \frac{2\left( \frac{3}{2} \right)_n}{(a + 2)(a + \frac{5}{2})_n} \left( 1 - 2\psi_1 + \frac{2n}{a + 3} (1 - 4\psi_1 + 6\psi_2 - 4\psi_3) \right). \tag{53}
\]

Finally, we give some evaluations when \( m \neq p \). When \( m = 2 \) and \( p = 0, \pm 1 \), for example, we have

\[
3F2 \left[ \frac{-2n, a, \; d + 2}{2a, \; d} \right] = \frac{\left( \frac{1}{2} \right)_n}{(a + \frac{1}{2})_n} \left( 1 + \frac{4n}{a} (\psi_1 + \psi_2) + \frac{8n(n - 1)}{a(a + 1)} \psi_2 \right), \tag{54}
\]

\[
3F2 \left[ \frac{-2n - 1, a, \; d + 2}{2a, \; d} \right] = -\frac{2\left( \frac{3}{2} \right)_n}{a(a + \frac{1}{2})_n} \left( \psi_1 + \frac{2n\psi_2}{a + 1} \right), \tag{55}
\]

\[
3F2 \left[ \frac{-2n, a, \; d + 2}{2a + 1, \; d} \right] = \frac{\left( \frac{1}{2} \right)_n}{(a + \frac{1}{2})_n} \left( 1 + \frac{4n\psi_2}{a + 1} \right), \tag{56}
\]

\[
3F2 \left[ \frac{-2n - 1, a, \; d + 2}{2a + 1, \; d} \right] = \frac{\left( \frac{3}{2} \right)_n}{(2a + 1)(a + \frac{1}{2})_n} \left( 1 - 4\psi_1 - \frac{4n\psi_2}{a + 1} \right). \tag{57}
\]
where $j$

Kummer’s second theorem applied to the confluent hypergeometric function $\, _1F_1$ is \cite[p. 12]{15}

\begin{equation}
3F_2 \left[ \frac{-2n, a, d + 2}{2a - 1, d}; 2 \right] = \frac{(\frac{1}{2})_n}{(a - \frac{1}{2})_n} \left\{ 1 + \frac{4n}{a} (2\varphi_1 + \varphi_2) + \frac{16n(n - 1)}{a(a + 1)} \varphi_2 \right\}, \tag{58}
\end{equation}

and

\begin{equation}
3F_2 \left[ \frac{-2n - 1, a, d + 2}{2a - 1, d}; 2 \right] = -\frac{(\frac{3}{2})_n}{(2a - 1)(a + \frac{1}{2})_n} \left\{ 1 + 4\varphi_1 + \frac{4n}{a} (2\varphi_1 + 3\varphi_2) + \frac{16n(n - 1)}{a(a + 1)} \varphi_2 \right\}. \tag{59}
\end{equation}

5. A second application of Theorem 1 to $\, _1F_1(x)$

Kummer’s second theorem applied to the confluent hypergeometric function $\, _1F_1$ is \cite[p. 12]{15}

\begin{equation}
e^{-x/2} \, _1F_1 \left[ \frac{a}{2a} : x \right] = aF_1 \left[ \frac{-}{a + \frac{1}{2}, \frac{x^2}{16}} \right] = (\frac{1}{4} x)^{a-q/2} \Gamma(a + \frac{1}{2}) I_{a-\frac{1}{2}}(\frac{1}{2} x), \tag{60}
\end{equation}

where $I_\nu(z)$ denotes a modified Bessel function of the first kind. We now show how the result in Theorem 1 can be used to derive a generalization of the above theorem for the functions

\begin{equation}
e^{-x/2} \, _1F_1 \left[ \frac{a}{2a \pm j} : x \right]
\end{equation}

for arbitrary integer $j$.

We have upon series expansion

\begin{equation}
e^{-x/2} \, _1F_1 \left[ \frac{a}{2a \pm j} : x \right] = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2} x)^n}{n!} \sum_{m=0}^{\infty} \frac{(a)_m}{(2a \pm j)_m} \frac{x^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (a)_m x^{m+n}}{2n(2a \pm j)_m m! n!}.
\end{equation}

Making the change of summation index $n \to n - m$ and using the fact that $(n - m)! = (-1)^m m! / (n)_m$, we find

\begin{equation}
e^{-x/2} \, _1F_1 \left[ \frac{a}{2a \pm j} : x \right] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (a)_m (n)_m x^n}{2n(2a \pm j)_m m! n!} = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2} x)^n}{n!} 2F_1 \left[ \frac{-n, a}{2a \pm j} : 2 \right].
\end{equation}

Separation of the above sum into even and odd $n$ and use of the evaluation of the $2F_1(2)$ series given in (6) and (7) followed by inversion of the order of summation leads to the result in the following theorem.

**Theorem 4** For integer $j$ we have

\begin{equation}
e^{-x/2} \, _1F_1 \left[ \frac{a}{2a \pm j} : x \right] = \sum_{r=0}^{j_0} (-1)^r \left( \frac{j}{2r} \right) \sum_{n=0}^{\infty} \frac{(-n)_r (a + \delta_j)_{n-r} x^{2n}}{2^{2n} (2a \pm j)_{2n} n!} + \sum_{r=0}^{j_0} (-1)^r \left( \frac{j}{2r + 1} \right) \sum_{n=0}^{\infty} \frac{(-n)_r (a + \delta_j)_{n-r} x^{2n+1}}{2^{2n+1} (2a \pm j)_{2n+1} n!}, \tag{61}
\end{equation}

where $j_0$ and $\delta_j$ are defined in Theorem 1.
When \( j = 0 \) it is easily seen that (61) reduces to (60).

The result (61) is given in a different form in terms of modified Bessel functions in [11, Vol. 3, p. 579]. For some interesting and general expressions for \( _2F_1(z) \) we refer to [1, pp. 563–564, Eq. (1)–(7)].

An alternative method of proof of this relation can be obtained from the following generalization of a result due to Ramanujan, considered in [7] for \( j = 0, \pm 1, \ldots, \pm 5 \).

**Theorem 5** Let \( \phi(t) \) be analytic for \( |t - 1| < R \), where \( R > 1 \). Suppose that \( a \) and \( \phi(t) \) are such that the order of summation in

\[
\sum_{n=0}^{\infty} \frac{2^n(a)_n\phi^{(n)}(0)}{(2a \pm j)_nn!} = \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(1)}{(2a \pm j)_{2n}n!} \sum_{r=0}^{j_0} (-1)^r \left( \frac{j}{2r} \right) (-n)_r(a + \delta_j)_{n-r}
\]

may be inverted. Then

\[
\sum_{n=0}^{\infty} \frac{\phi^{(2n+1)}(1)}{(2a \pm j)_{2n+1}n!} \sum_{r=0}^{j_0} (-1)^r \left( \frac{j}{2r + 1} \right) (-n)_r(a + \delta_j)_{n-r}
\]

for \( j = 0, 1, 2, \ldots \), where \( j_0 \) and \( \delta_j \) are defined in Theorem 1.

**Proof** We employ the identity obtained in [7, §2] relating the sum of derivatives of \( \phi(t) \) evaluated at \( t = 0 \) to a sum of derivatives evaluated at \( t = 1 \) given by

\[
\sum_{n=0}^{\infty} \frac{2^n(a)_n\phi^{(n)}(0)}{(2a \pm j)_nn!} = \sum_{n=0}^{\infty} \frac{(-1)^n\phi^{(n)}(1)}{n!} {}_2F_1 \left[ -n, a \atop 2a \pm j \right].
\]

Separation of the sum on the right-hand side into even and odd \( n \) and insertion of the \( {}_2F_1(2) \) series evaluations in (6) and (7) then establishes the theorem. \( \square \)

If we take \( \phi(t) = e^{xt/2} \) in (62), where \( x \) is an arbitrary variable, we immediately obtain (61) after reversal of the order of summation. Other choices for \( \phi(t) \) in (62) were considered in [7] but with the integer \( j \) restricted to the values \( j = 0, \pm 1, \ldots, \pm 5 \).

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**References**

