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Evaluations of some terminating hypergeometric ${}_2F_1(2)$ series with applications

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Abstract: Explicit expressions for the hypergeometric series ${}_2F_1(-n, a; 2a \pm j; 2)$ and ${}_2F_1(-n, a; -2n \pm j; 2)$ for positive integer n and arbitrary integer j are obtained with the help of generalizations of Kummer's second and third summation theorems obtained earlier by Rakha and Rathie. Results for $|j| \leq 5$ derived previously using different methods are also obtained as special cases. Two applications are considered, where the first summation formula is applied to a terminating ${}_3F_2(2)$ series and the confluent hypergeometric function ${}_1F_1(x)$.

Key words: Terminating hypergeometric series, generalized Kummer's second and third summation theorems

1. Introduction

In a problem arising in a model of a biological problem, Samoletov [14] obtained by means of a mathematical induction argument the following sum containing factorials:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \frac{(2k+1)!!}{k!(k+1)!} = \frac{(-1)^n}{\sqrt{n!(n+1)!}} \left(\sqrt{n+1} \, \frac{(n-1)!!}{n!!} \right)^{(-1)^n},$$

where throughout n denotes a positive integer and, as usual,

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \qquad (2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n n!}.$$

Samoletov [14] also expressed the above sum in the equivalent hypergeometric form:

$${}_2F_1\left[\begin{matrix} -n, \frac{3}{2} \\ 2 \end{matrix}; 2\right] = \begin{cases} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}n+1)}, & (n \text{ even}) \\ \frac{-\Gamma(\frac{1}{2}n+1)}{\sqrt{\pi}\Gamma(\frac{1}{2}n+\frac{3}{2})}, & (n \text{ odd}). \end{cases}$$

Subsequently, Srivastava [17] pointed out that this result could be easily derived from a known hypergeometric summation formula [11, Vol. 2, p. 493] for ${}_2F_1(-n, a; 2a-1; 2)$ with $a = \frac{3}{2}$, which is a contiguous result to the

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well-known summation

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a\end{matrix}; 2\right] = \frac{2^n\sqrt{\pi}\,\Gamma(1-a)}{(2a)_n\Gamma(\frac{1}{2}-\frac{1}{2}n)\Gamma(1-a-\frac{1}{2}n)} \qquad (n = 0, 1, 2, \ldots). \tag{1} $$

The aim in this note is to obtain explicit expressions for

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a\pm j\end{matrix}; 2\right] \qquad \text{and} \qquad {}_2F_1\left[\begin{matrix}-n, a\\ -2n\pm j\end{matrix}; 2\right] \tag{2} $$

for arbitrary integer j. We shall employ the following generalizations of Kummer's second and third summation theorems given in [13] (we correct a misprint in Theorem 6 of this reference). These are respectively

$$ {}_2F_1\left[\begin{matrix}\alpha, \beta\\ \frac{1}{2}(\alpha+\beta\pm j+1)\end{matrix}; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\pm\frac{1}{2}j)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})}\,\frac{\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\mp\frac{1}{2}j)}{\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}+\frac{1}{2}j)} $$

$$ \times\sum_{r=0}^{j}(\mp 1)^r\binom{j}{r}\frac{(\frac{1}{2}\beta)_{r/2}}{(\frac{1}{2}\alpha+\frac{1}{2})_{(r-j)/2}} \tag{3} $$

and

$$ {}_2F_1\left[\begin{matrix}\alpha, 1-\alpha\pm j\\ \gamma\end{matrix}; \frac{1}{2}\right] = \frac{2^{\pm j}\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2}\gamma+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\gamma-\frac{1}{2}\alpha+\frac{1}{2})}\,\frac{\Gamma(\alpha\mp j)}{\Gamma(\alpha+\epsilon_j)} $$

$$ \times\sum_{r=0}^{j}(\mp 1)^r\binom{j}{r}\frac{(\frac{1}{2}\gamma-\frac{1}{2}\alpha)_{r/2}}{(\frac{1}{2}\gamma+\frac{1}{2}\alpha)_{r/2-\delta_j}} \tag{4} $$

for $j = 0, 1, 2, \ldots$, where $\epsilon_j = 0$ (resp. j), $\delta_j = j$ (resp. 0) when the upper (resp. lower) signs are taken and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol defined for *arbitrary* index k. The formulas (3) (for $j > 0$) and (4) (separately for $j > 0$ and $j < 0$) are given in a different form in [1, p. 582, (130)–(132)]. When $j = 0$, the summations (3) and (4) reduce to the well-known second and third summation theorems due to Kummer [16, p. 243]:

$$ {}_2F_1\left[\begin{matrix}\alpha, \beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix}; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})} $$

and*

$$ {}_2F_1\left[\begin{matrix}\alpha, 1-\alpha\\ \gamma\end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2}\gamma+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\gamma-\frac{1}{2}\alpha+\frac{1}{2})}. $$

In addition, we shall make use of the transformation [10, (15.8.6)]

$$ {}_2F_1\left[\begin{matrix}-n, \beta\\ \gamma\end{matrix}; 2\right] = \frac{(-2)^n(\beta)_n}{(\gamma)_n}\,{}_2F_1\left[\begin{matrix}-n, 1-\gamma-n\\ 1-\beta-n\end{matrix}; \frac{1}{2}\right]. \tag{5} $$

Expressions for the series in (2) for arbitrary integer j were recently obtained by Chu [3] using a different approach. This involved expressing the series for $j \neq 0$ as finite sums of ${}_2F_1(2)$ series in (2) with $j = 0$. The

*In [16, p. 243], this summation formula is referred to as Bailey's theorem. However, it has been pointed out in [2] that this theorem was originally found by Kummer.

cases with $|j| \leq 5$ were also given by Kim and Rathie [6] and Kim et al. [5]. An application of the first series in (2) for $j = 0, 1, \ldots, 5$ was discussed in [7]. In Sections 4 and 5, we give two additional applications, the first to the evaluation of a terminating ${}_3F_2(2)$ series and the second to the confluent hypergeometric function ${}_1F_1(x)$.

2. Statement of the results

Our principal results are stated in the following two theorems.

Theorem 1 *Let n be a positive integer, let a be a complex parameter, and define $j_0 = \lfloor \frac{1}{2} j \rfloor$. Then we have*

$$ {}_2F_1\left[\begin{array}{c} -2n, a \\ 2a \pm j \end{array}; 2\right] = \frac{2^{2n}(\frac{1}{2})_n}{(2a \pm j)_{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a + \delta_j)_{n-r} \tag{6} $$

and

$$ {}_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a \pm j \end{array}; 2\right] = \frac{\pm 2^{2n}(\frac{3}{2})_n}{(2a \pm j)_{2n+1}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r+1} (-n)_r (a + \delta_j)_{n-r} \tag{7} $$

for $j = 0, 1, 2, \ldots$, where $\delta_j = j$ (resp. 0) when the upper (resp. lower) signs are taken.

Proof From the result (5) we have

$$ {}_2F_1\left[\begin{array}{c} -n, a \\ 2a \pm j \end{array}; 2\right] = \frac{(-2)^n (a)_n}{(2a \pm j)_n} \, {}_2F_1\left[\begin{array}{c} -n, 1 - 2a \mp j - n \\ 1 - a - n \end{array}; \frac{1}{2}\right]. $$

The hypergeometric function on the right-hand side can be summed by the generalized second Kummer summation theorem (3). Proceeding first with the function whose denominator parameter is $2a + j$, we put $\alpha = 1 - 2a - j - n$ and $\beta = -n$ in (3) to find, after some straightforward algebra,

$$ {}_2F_1\left[\begin{array}{c} -n, a \\ 2a + j \end{array}; 2\right] = \frac{(-2)^n (a)_n}{(2a + j)_n} \frac{\sqrt{\pi}\,\Gamma(1 - a - n)}{\Gamma(-\frac{1}{2}n + \frac{1}{2})\Gamma(1 - a)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{(-\frac{1}{2}n)_{r/2}}{(1 - a - j)_{(r-n)/2}} $$

$$ = \frac{2^n}{(2a + j)_n} \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{2}n + \frac{1}{2})} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{(-\frac{1}{2}n)_{r/2}}{(1 - a - j)_{(r-n)/2}} \,. \tag{8} $$

Now replace n by $2n$ and use the reflection formula for the gamma function to yield

$$ {}_2F_1\left[\begin{array}{c} -2n, a \\ 2a + j \end{array}; 2\right] = \frac{(-1)^n 2^{2n}(\frac{1}{2})_n}{(2a + j)_{2n}} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{(-n)_{r/2}}{(1 - a - j)_{r/2-n}} \,. $$

Since $(-n)_{r/2}$ vanishes for odd r and positive integer n, we finally obtain

$$ {}_2F_1\left[\begin{array}{c} -2n, a \\ 2a + j \end{array}; 2\right] = \frac{2^{2n}(\frac{1}{2})_n}{(2a + j)_{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a + j)_{n-r} \,, \tag{9} $$

where $j_0 = \lfloor \frac{1}{2} j \rfloor$ and we have used the fact that $(1 - a - j)_{r-n} = (-1)^{n-r}/(a + j)_{n-r}$.

If we replace n by $2n+1$ in (8), we find

$$ {}_2F_1\left[\begin{array}{c}-2n-1,a\\ 2a+j\end{array};2\right]=\frac{2^{2n+1}}{(2a+j)_{2n+1}}\,\frac{\sqrt{\pi}}{\Gamma(-n-\frac{1}{2})}\sum_{r=0}^{j_0}(-1)^r\binom{j}{r}\frac{(-n)_{(r-1)/2}}{(1-a-j)_{(r/2-1/2-n}}, $$

where we have used

$$ \frac{(-n-\frac{1}{2})_{r/2}}{\Gamma(-n)}=\frac{(-n)_{(r-1)/2}}{\Gamma(-n-\frac{1}{2})}. $$

Since $(-n)_{(r-1)/2}$ vanishes for even r and positive integer n, we find

$$ {}_2F_1\left[\begin{array}{c}-2n-1,a\\ 2a+j\end{array};2\right]=\frac{2^{2n}(\frac{3}{2})_n}{(2a+j)_{2n+1}}\sum_{r=0}^{j_0}(-1)^r\binom{j}{2r+1}(-n)_r(a+j)_{n-r}. \tag{10} $$

Proceeding in a similar manner for the function whose denominator parameter is $2a-j$, we have upon letting $\alpha=1-a+j-n$ and $\beta=-n$ in (3)

$$ {}_2F_1\left[\begin{array}{c}-n,a\\ 2a-j\end{array};2\right]=\frac{2^n}{(2a-j)_n}\,\frac{\sqrt{\pi}}{\Gamma(-\frac{1}{2}n+\frac{1}{2})}\sum_{r=0}^{j}\binom{j}{r}\frac{(-\frac{1}{2}n)_{r/2}}{(1-a)_{(r-n)/2}}. \tag{11} $$

Then we have

$$ {}_2F_1\left[\begin{array}{c}-2n,a\\ 2a-j\end{array};2\right]=\frac{2^{2n}(\frac{1}{2})_n}{(2a-j)_{2n}}\sum_{r=0}^{j_0}(-1)^r\binom{j}{2r}(-n)_r(a)_{n-r} \tag{12} $$

and

$$ {}_2F_1\left[\begin{array}{c}-2n-1,a\\ 2a-j\end{array};2\right]=\frac{-2^{2n}(\frac{3}{2})_n}{(2a-j)_{2n+1}}\sum_{r=0}^{j_0}(-1)^r\binom{j}{2r+1}(-n)_r(a)_{n-r}. \tag{13} $$

The summations in (9), (10) and (12), (13) then correspond to the results stated in the theorem. □

We remark that in (7) the upper limit of summation can be replaced by $\lfloor\frac{1}{2}j\rfloor-1$ when j is even. Also, since $(-n)_r$ vanishes when $r>n$, it is possible to replace the upper summation limit in both (6) and (7) by n whenever $n>\lfloor\frac{1}{2}j\rfloor$.

Theorem 2 *Let n be a positive integer and a be a complex parameter. Then we have*

$$ {}_2F_1\left[\begin{array}{c}-n,a\\ -2n+j\end{array};2\right]=\frac{2^{2n-j}(n-j)!}{(2n-j)!}\sum_{r=0}^{j}\binom{j}{r}(\tfrac{1}{2}a+\tfrac{1}{2}-\tfrac{1}{2}r)_n \tag{14} $$

provided that j does not lie in the interval $[n+1,2n]$ (where the hypergeometric function on the left-hand side of (14) is, in general, not defined), and

$$ {}_2F_1\left[\begin{array}{c}-n,a\\ -2n-j\end{array};2\right]=\frac{2^{2n+j}n!}{(2n+j)!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}(\tfrac{1}{2}a+\tfrac{1}{2}-\tfrac{1}{2}r)_{n+j} \tag{15} $$

for $j=0,1,2,\ldots$. When $j\geq 2n+1$ in (14), the ratio of factorials $(n-j)!/(2n-j)!$ can be replaced by $(-1)^n(j-2n-1)!/(j-n-1)!$.

Proof From the result (5) we have

$$ {}_2F_1\left[\begin{matrix} -n, a \\ -2n \mp j \end{matrix}; 2\right] = \frac{2^n (a)_n (n \pm j)!}{(2n \pm j)!} \, {}_2F_1\left[\begin{matrix} -n, 1+n \pm j \\ 1-a-n \end{matrix}; \frac{1}{2}\right] $$

when the parameters are such that the hypergeometric functions make sense. The hypergeometric function on the right-hand side can be summed by the generalized third Kummer theorem (4), where we put $\alpha = -n$ and $\gamma = 1 - a - n$. Then we obtain

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -n, a \\ -2n + j \end{matrix}; 2\right] &= \frac{2^{n-j} (a)_n (n-j)!}{(2n-j)!} \, \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma + \frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{1}{2}a)\Gamma(1 - \frac{1}{2}a)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}r)}{\Gamma(\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}r - n)} \\ &= \frac{(-1)^n 2^{2n-j} (a)_n (n-j)!}{(2n-j)!} \, \frac{\Gamma(1-a-n)}{\Gamma(1-a)} \sum_{r=0}^{j} \binom{j}{r} (\tfrac{1}{2} - \tfrac{1}{2}a + \tfrac{1}{2}r)_n \\ &= \frac{2^{2n-j} (n-j)!}{(2n-j)!} \sum_{r=0}^{j} \binom{j}{r} (\tfrac{1}{2} - \tfrac{1}{2}a + \tfrac{1}{2}r)_n, \end{aligned} $$

where we have used the reflection formula for the gamma function and

$$ (2a)_{2n} = 2^{2n} (a)_n (a + \tfrac{1}{2})_n. $$

Similarly, we find

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -n, a \\ -2n - j \end{matrix}; 2\right] &= \frac{(-1)^j 2^{n+j} (a)_n n!}{(2n+j)!} \, \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma + \frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{1}{2}a)\Gamma(1 - \frac{1}{2}a)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}r)}{\Gamma(\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}r - n - j)} \\ &= \frac{(-1)^n 2^{2n+j} (a)_n n!}{(2n+j)!} \, \frac{\Gamma(1-a-n)}{\Gamma(1-a)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (\tfrac{1}{2} - \tfrac{1}{2}a + \tfrac{1}{2}r)_{n+j} \\ &= \frac{2^{2n+j} n!}{(2n+j)!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (\tfrac{1}{2} - \tfrac{1}{2}a + \tfrac{1}{2}r)_{n+j}, \end{aligned} $$

which establishes the theorem. □

The sums on the right-hand sides of (14) and (15) can be written in an alternative form involving just two Pochhammer symbols containing the index n by making use of the result

$$ (\alpha - r)_n = \frac{(\alpha)_n (1-\alpha)_r}{(1 - \alpha - n)_r} $$

for positive integers r and n. Then we find, with $j_0 = \lfloor \frac{1}{2} j \rfloor$,

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} -n, a \\ -2n + j \end{matrix}; 2\right] \\ &= \frac{2^{2n-j} (n-j)!}{(2n-j)!} \left\{ (\tfrac{1}{2}a + \tfrac{1}{2})_n \sum_{r=0}^{j_0} \binom{j}{2r} A_r(n,0) + (\tfrac{1}{2}a)_n \sum_{r=0}^{j_0} \binom{j}{2r+1} B_r(n,0) \right\} \qquad (16) \end{aligned} $$

and

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n-j\end{matrix};2\right]
$$

$$
=\frac{2^{2n+j}n!}{(2n+j)!}\left\{(\tfrac{1}{2}a+\tfrac{1}{2})_{n+j}\sum_{r=0}^{j_0}\binom{j}{2r}A_r(n,j)-(\tfrac{1}{2}a)_{n+j}\sum_{r=0}^{j_0}\binom{j}{2r+1}B_r(n,j)\right\}, \tag{17}
$$

where

$$
A_r(n,j):=\frac{(\frac{1}{2}-\frac{1}{2}a)_r}{(\frac{1}{2}-\frac{1}{2}a-n-j)_r}, \qquad B_r(n,j):=\frac{(1-\frac{1}{2}a)_r}{(1-\frac{1}{2}a-n-j)_r}.
$$

Again, when j is even, the upper summation limit in the second sums in (16) and (17) can be replaced by j_0-1, if so desired.

3. Special cases

If we set $0 \le j \le 5$ in (6) we obtain the following summations:

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}={}_2F_1\left[\begin{matrix}-2n, a\\ 2a+1\end{matrix};2\right], \tag{18}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a+2\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{3}{2})_n}\left(1+\frac{2n}{a+1}\right), \tag{19}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a+3\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{3}{2})_n}\left(1+\frac{4n}{a+2}\right), \tag{20}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a+4\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{5}{2})_n}\left(1+\frac{8n}{a+2}+\frac{8n(n-1)}{(a+2)(a+3)}\right), \tag{21}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a+5\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{5}{2})_n}\left(1+\frac{12n}{a+3}+\frac{16n(n-1)}{(a+3)(a+4)}\right) \tag{22}
$$

and

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a-1\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a-\frac{1}{2})_n}, \tag{23}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a-2\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a-\frac{1}{2})_n}\left(1+\frac{2n}{a-1}\right), \tag{24}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a-3\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a-\frac{3}{2})_n}\left(1+\frac{4n}{a-1}\right), \tag{25}
$$

$$
{}_2F_1\left[\begin{matrix}-2n, a\\ 2a-4\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a-\frac{3}{2})_n}\left(1+\frac{8n}{a-2}+\frac{8n(n-1)}{(a-1)(a-2)}\right), \tag{26}
$$

$$ {}_2F_1\left[\begin{matrix} -2n, a \\ 2a-5 \end{matrix};2\right] = \frac{(\frac{1}{2})_n}{(a-\frac{5}{2})_n}\left(1+\frac{12n}{a-2}+\frac{16n(n-1)}{(a-1)(a-2)}\right). \tag{27} $$

Similarly, if we set $0 \leq j \leq 5$ in (7) we obtain the following summations:

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a \end{matrix};2\right] = 0, \tag{28} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a+1 \end{matrix};2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}, \tag{29} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a+2 \end{matrix};2\right] = \frac{2(\frac{3}{2})_n}{(2a+2)(a+\frac{3}{2})_n}, \tag{30} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a+3 \end{matrix};2\right] = \frac{(\frac{3}{2})_n}{(2a+3)(a+\frac{5}{2})_n}\left(3+\frac{4n}{a+2}\right), \tag{31} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a+4 \end{matrix};2\right] = \frac{(\frac{3}{2})_n}{(2a+4)(a+\frac{5}{2})_n}\left(4+\frac{8n}{a+3}\right), \tag{32} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a+5 \end{matrix};2\right] = \frac{(\frac{3}{2})_n}{(2a+5)(a+\frac{7}{2})_n}\left(5+\frac{20n}{a+3}+\frac{16n(n-1)}{(a+3)(a+4)}\right) \tag{33} $$

and

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a-1 \end{matrix};2\right] = -\frac{(\frac{3}{2})_n}{(2a-1)(a+\frac{1}{2})_n}, \tag{34} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a-2 \end{matrix};2\right] = -\frac{2(\frac{3}{2})_n}{(2a-2)(a-\frac{1}{2})_n}, \tag{35} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a-3 \end{matrix};2\right] = -\frac{(\frac{3}{2})_n}{(2a-3)(a-\frac{1}{2})_n}\left(3+\frac{4n}{a-1}\right), \tag{36} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a-4 \end{matrix};2\right] = -\frac{(\frac{3}{2})_n}{(2a-4)(a-\frac{3}{2})_n}\left(4+\frac{8n}{a-1}\right), \tag{37} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a-5 \end{matrix};2\right] = -\frac{(\frac{3}{2})_n}{(2a-5)(a-\frac{3}{2})_n}\left(5+\frac{20n}{a-2}+\frac{16n(n-1)}{(a-1)(a-2)}\right). \tag{38} $$

Finally, from (16) and (17) we obtain:

$$ {}_2F_1\left[\begin{matrix} -n, a \\ -2n \end{matrix};2\right] = \frac{2^{2n}n!}{(2n)!}(\tfrac{1}{2}a+\tfrac{1}{2})_n = \frac{(\frac{1}{2}a+\frac{1}{2})_n}{(\frac{1}{2})_n}, \tag{39} $$

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n+1\end{matrix};2\right]=\frac{2^{2n-1}(n-1)!}{(2n-1)!}\{(\tfrac{1}{2}a+\tfrac{1}{2})_n+(\tfrac{1}{2}a)_n\}, \tag{40}
$$

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n-1\end{matrix};2\right]=\frac{2^{2n+1}n!}{(2n+1)!}(\tfrac{1}{2}a+\tfrac{1}{2})_{n+1}, \tag{41}
$$

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n+2\end{matrix};2\right]=\frac{2^{2n-1}(n-2)!}{(2n-2)!}\left\{\frac{1-a-n}{1-a-2n}(\tfrac{1}{2}a+\tfrac{1}{2})_n+(\tfrac{1}{2}a)_n\right\}, \tag{42}
$$

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n-2\end{matrix};2\right]=\frac{2^{2n+3}n!}{(2n+2)!}\left\{\frac{(1-a-n-j)}{1-a-2n-2j}(\tfrac{1}{2}a+\tfrac{1}{2})_{n+2}-(\tfrac{1}{2}a)_{n+2}\right\} \tag{43}
$$

and so on.

The above evaluations agree with those given in [3, 6], although presented in a different format; the results (18) and (23), together with (28), (29), and (34), are also recorded in [11] in another form.

4. An application of Theorem 1 to a terminating ${}_3F_2(2)$ series

From the results in Section 2 we shall establish a summation formula for the terminating ${}_3F_2(2)$ series

$$
{}_3F_2\left[\begin{matrix}-n, a, & d+m\\ 2a+p, & d\end{matrix};2\right],
$$

where m, n are positive integers and p is an arbitrary integer such that $2a+p\neq 0,-1,-2,\ldots,-n+1$. The cases $m=1$ and $m=2$ (when $m=p$) have been obtained previously by different means in [4, 12]. When $m=p=0$ the above series reduces to that in (1).

We employ a general result proved in [9, Lemma 4] expressing an ${}_{r+2}F_{r+1}(x)$ hypergeometric function, with r pairs of numeratorial and denominatorial parameters differing by positive integers, as a finite sum of ${}_2F_1(x)$ functions. In the particular case $r=1$, we obtain for positive integer m

$$
{}_3F_2\left[\begin{matrix}a, b & d+m\\ c & d\end{matrix};x\right]=\frac{1}{(d)_m}\sum_{j=0}^{m}A_j\frac{(a)_j(b)_j}{(c)_j}x^j{}_2F_1\left[\begin{matrix}a+j, b+j\\ c+j\end{matrix};x\right]. \tag{44}
$$

The coefficients A_j are defined by

$$
A_j=\sum_{k=j}^{m}\sigma_{m-k}\mathbf{S}_k^{(j)},
$$

where $\mathbf{S}_k^{(j)}$ is the Stirling number of the second kind and the σ_j $(0\leq j\leq m)$ are generated by the relation

$$
(d+x)_m=\sum_{j=0}^{m}\sigma_{m-j}x^j.
$$

It was shown in [8] that

$$
A_j=\frac{(-1)^j(d)_m}{j!}{}_2F_1\left[\begin{matrix}-j, d+m\\ d\end{matrix};1\right]=\frac{(-1)^j(-m)_j(d)_m}{j!\,(d)_j}, \tag{45}
$$

the second result following by application of Vandermonde's theorem [16, p. 243].

In order to simplify the presentation of our summation theorem we introduce the coefficients $C_r(n,j)$ given by

$$C_r(n,j)=\frac{\Gamma(-\frac{1}{2}n+\frac{1}{2}j+\frac{1}{2}r)}{\Gamma(-\frac{1}{2}n+\frac{1}{2}j)\Gamma(-\frac{1}{2}n+\frac{1}{2}j+\frac{1}{2})},$$

which yield the even- and odd-order values

$$C_{2r}(n,j)=\frac{(-\frac{1}{2}n+\frac{1}{2}j)_r}{\Gamma(-\frac{1}{2}n+\frac{1}{2}j+\frac{1}{2})},\qquad C_{2r+1}(n,j)=\frac{(-\frac{1}{2}n+\frac{1}{2}j+\frac{1}{2})_r}{\Gamma(-\frac{1}{2}n+\frac{1}{2}j)}.$$

We further introduce the finite sums for $0\le j\le m$:

$$S_j^{(1)}(n,p)=\sum_{r=0}^{p-j}\binom{p-j}{r}\frac{(-1)^r C_r(n,j)}{(1-a-p)_{(r-n+j)/2}}\quad (p\ge j),$$

$$S_j^{(2)}(n,p)=\sum_{r=0}^{j-p}\binom{j-p}{r}\frac{C_r(n,j)}{(1-a-j)_{(r-n+j)/2}}\quad (p\le j).$$

Then we obtain the following theorem.

Theorem 3 *Let m, n be positive integers and p be an arbitrary integer such that $2a+p\neq 0,-1,-2,\ldots,-n+1$. Then*

$$ {}_3F_2\left[\begin{matrix}-n,a, & d+m\\ 2a+p, & d\end{matrix};2\right]=\begin{cases}\dfrac{2^n\sqrt{\pi}}{(2a+p)_n}\displaystyle\sum_{j=0}^{m}\frac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j}S_j^{(1)}(n,p) & (p\ge m>0)\\ \dfrac{2^n\sqrt{\pi}}{(2a+p)_n}\left\{\displaystyle\sum_{j=0}^{p}\frac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j}S_j^{(1)}(n,p)\right.\\ \qquad\left.+\displaystyle\sum_{j=p+1}^{m}\frac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j}S_j^{(2)}(n,p)\right\} & (0\le p<m)\\ \dfrac{2^n\sqrt{\pi}}{(2a+p)_n}\displaystyle\sum_{j=0}^{m}\frac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j}S_j^{(2)}(n,p) & (p\le 0).\end{cases}$$

Proof If we put $b=-n$, $c=2a+p$ and $x=2$ in (44), combined with (45), we find

$$ {}_3F_2\left[\begin{matrix}-n,a & d+m\\ 2a+p & d\end{matrix};2\right]=\sum_{j=0}^{m}\frac{(-2)^j(-m)_j(-n)_j(a)_j}{j!\,(2a+p)_j(d)_j}\,{}_2F_1\left[\begin{matrix}-n+j,a+j\\ 2a+p+j\end{matrix};2\right].$$

Substitution of (8) and (11) with $n\to n-j$, $a\to a+j$ and the parameter in the denominator replaced by $2(a+j)+p-j$, followed by some straightforward algebra to distinguish between the cases with $p-j\ge 0$ and $p-j<0$, leads to the results stated in the theorem. □

We conclude this section by giving some specific examples of the summation in Theorem 3. Alternatively, these results may be obtained from (44), (45), and the ${}_2F_1(2)$ summations listed in Section 3. For convenience in presentation we define

$$\wp_k:=\frac{(a)_k}{(d)_k}\qquad (k=1,2,\ldots).$$

When $m = p$, we have for $1 \leq m \leq 4$ and positive integer n

$$ {}_3F_2\left[\begin{matrix} -2n, a, & d+1 \\ 2a+1, & d \end{matrix} ;2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, \tag{46} $$

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+1 \\ 2a+1, & d \end{matrix} ;2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}\left(1-2\wp_1\right), \tag{47} $$

$$ {}_3F_2\left[\begin{matrix} -2n, a, & d+2 \\ 2a+2, & d \end{matrix} ;2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{3}{2})_n}\left\{1+\frac{2n}{a+1}\left[1-2\wp_1+2\wp_2\right]\right\}, \tag{48} $$

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+2 \\ 2a+2, & d \end{matrix} ;2\right] = \frac{(\frac{3}{2})_n}{(a+1)(a+\frac{3}{2})_n}\left\{1-2\wp_1\right\}, \tag{49} $$

$$ {}_3F_2\left[\begin{matrix} -2n, a, & d+3 \\ 2a+3, & d \end{matrix} ;2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{3}{2})_n}\left\{1+\frac{4n}{a+2}(1-3\wp_1+3\wp_2)\right\}, \tag{50} $$

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+3 \\ 2a+3, & d \end{matrix} ;2\right] = \frac{(\frac{3}{2})_n}{(2a+3)(a+\frac{5}{2})_n}\left\{3-6\wp_1+\frac{4n}{a+2}(1-3\wp_1+3\wp_2-2\wp_3)\right\}, \tag{51} $$

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} -2n, a, & d+4 \\ 2a+4, & d \end{matrix} ;2\right] \\ &= \frac{(\frac{1}{2})_n}{(a+\frac{5}{2})_n}\left\{1+\frac{8n}{a+2}\left(1-3\wp_1+3\wp_2\right)+\frac{8n(n-1)}{(a+2)(a+3)}\left(1-4\wp_1+6\wp_2-4\wp_3+2\wp_4\right)\right\}, \end{aligned} \tag{52} $$

and

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+4 \\ 2a+4, & d \end{matrix} ;2\right] = \frac{2(\frac{3}{2})_n}{(a+2)(a+\frac{5}{2})_n}\left\{1-2\wp_1+\frac{2n}{a+3}\left(1-4\wp_1+6\wp_2-4\wp_3\right)\right\}. \tag{53} $$

Finally, we give some evaluations when $m \neq p$. When $m = 2$ and $p = 0, \pm 1$, for example, we have

$$ {}_3F_2\left[\begin{matrix} -2n, a, & d+2 \\ 2a, & d \end{matrix} ;2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}\left\{1+\frac{4n}{a}(\wp_1+\wp_2)+\frac{8n(n-1)}{a(a+1)}\wp_2\right\}, \tag{54} $$

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+2 \\ 2a, & d \end{matrix} ;2\right] = -\frac{2(\frac{3}{2})_n}{a(a+\frac{1}{2})_n}\left\{\wp_1+\frac{2n\wp_2}{a+1}\right\}, \tag{55} $$

$$ {}_3F_2\left[\begin{matrix} -2n, a, & d+2 \\ 2a+1, & d \end{matrix} ;2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}\left\{1+\frac{4n\wp_2}{a+1}\right\}, \tag{56} $$

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+2 \\ 2a+1, & d \end{matrix} ;2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}\left\{1-4\wp_1-\frac{4n\wp_2}{a+1}\right\}, \tag{57} $$

$$ {}_3F_2\left[\begin{matrix} -2n, a, & d+2 \\ 2a-1, & d \end{matrix} ;2\right] = \frac{(\frac{1}{2})_n}{(a-\frac{1}{2})_n}\left\{1+\frac{4n}{a}(2\wp_1+\wp_2)+\frac{16n(n-1)}{a(a+1)}\wp_2\right\}, \tag{58} $$

and

$$ {}_3F_2\left[\begin{matrix} -2n-1, a, & d+2 \\ 2a-1, & d \end{matrix} ;2\right] = -\frac{(\frac{3}{2})_n}{(2a-1)(a+\frac{1}{2})_n}\left\{1+4\wp_1+\frac{4n}{a}(2\wp_1+3\wp_2)+\frac{16n(n-1)}{a(a+1)}\wp_2\right\}. \tag{59} $$

5. A second application of Theorem 1 to ${}_1F_1(x)$

Kummer's second theorem applied to the confluent hypergeometric function ${}_1F_1$ is [15, p. 12]

$$ e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix} ;x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ;\frac{x^2}{16}\right] = (\tfrac{1}{4}x)^{\frac{1}{2}-a}\Gamma(a+\tfrac{1}{2})I_{a-\frac{1}{2}}(\tfrac{1}{2}x), \tag{60} $$

where $I_\nu(z)$ denotes a modified Bessel function of the first kind. We now show how the result in Theorem 1 can be used to derive a generalization of the above theorem for the functions

$$ e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a\pm j \end{matrix} ;x\right] $$

for arbitrary integer j.

We have upon series expansion

$$ e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a\pm j \end{matrix} ;x\right] = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2}x)^n}{n!}\sum_{m=0}^{\infty}\frac{(a)_m}{(2a\pm j)_m}\frac{x^m}{m!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n(a)_m x^{m+n}}{2^n(2a\pm j)_m m!n!}. $$

Making the change of summation index $n\to n-m$ and using the fact that $(n-m)! = (-1)^m m!/(-n)_m$, we find

$$ e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a\pm j \end{matrix} ;x\right] = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n(a)_m(-n)_m x^n}{2^{n-m}(2a\pm j)_m m!n!} $$

$$ = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2}x)^n}{n!}\,{}_2F_1\left[\begin{matrix} -n, a \\ 2a\pm j \end{matrix} ;2\right]. $$

Separation of the above sum into even and odd n and use of the evaluation of the ${}_2F_1(2)$ series given in (6) and (7) followed by inversion of the order of summation leads to the result in the following theorem.

Theorem 4 *For integer j we have*

$$ e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a\pm j \end{matrix} ;x\right] = \sum_{r=0}^{j_0}(-1)^r\binom{j}{2r}\sum_{n=0}^{\infty}\frac{(-n)_r(a+\delta_j)_{n-r}x^{2n}}{2^{2n}(2a\pm j)_{2n}n!} $$

$$ \mp\sum_{r=0}^{j_0}(-1)^r\binom{j}{2r+1}\sum_{n=0}^{\infty}\frac{(-n)_r(a+\delta_j)_{n-r}x^{2n+1}}{2^{2n+1}(2a\pm j)_{2n+1}n!}, \tag{61} $$

where j_0 and δ_j are defined in Theorem 1.

When $j = 0$ it is easily seen that (61) reduces to (60).

The result (61) is given in a different form in terms of modified Bessel functions in [11, Vol. 3, p. 579]. For some interesting and general expressions for ${}_2F_1(z)$ we refer to [1, pp. 563–564, Eq. (1)–(7)].

An alternative method of proof of this relation can be obtained from the following generalization of a result due to Ramanujan, considered in [7] for $j = 0, \pm 1, \ldots, \pm 5$.

Theorem 5 *Let $\phi(t)$ be analytic for $|t-1| < R$, where $R > 1$. Suppose that a and $\phi(t)$ are such that the order of summation in*

$$\sum_{n=0}^{\infty} \frac{2^n (a)_n}{(2a \pm j)_n n!} \sum_{k=n}^{\infty} \frac{(-1)^k}{k!} (-k)_n \phi^{(k)}(1)$$

may be inverted. Then

$$\sum_{n=0}^{\infty} \frac{2^n (a)_n \phi^{(n)}(0)}{(2a \pm j)_n n!} = \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(1)}{(2a \pm j)_{2n} n!} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a+\delta_j)_{n-r}$$

$$\mp \sum_{n=0}^{\infty} \frac{\phi^{(2n+1)}(1)}{(2a \pm j)_{2n+1} n!} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r+1} (-n)_r (a+\delta_j)_{n-r} \tag{62}$$

for $j = 0, 1, 2, \ldots$, where j_0 and δ_j are defined in Theorem 1.

Proof We employ the identity obtained in [7, §2] relating the sum of derivatives of $\phi(t)$ evaluated at $t = 0$ to a sum of derivatives evaluated at $t = 1$ given by

$$\sum_{n=0}^{\infty} \frac{2^n (a)_n \phi^{(n)}(0)}{(2a \pm j)_n n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{(n)}(1)}{n!} \, {}_2F_1 \left[\begin{matrix} -n, a \\ 2a \pm j \end{matrix} ; 2 \right].$$

Separation of the sum on the right-hand side into even and odd n and insertion of the ${}_2F_1(2)$ series evaluations in (6) and (7) then establishes the theorem. □

If we take $\phi(t) = e^{xt/2}$ in (62), where x is an arbitrary variable, we immediately obtain (61) after reversal of the order of summation. Other choices for $\phi(t)$ in (62) were considered in [7] but with the integer j restricted to the values $j = 0, \pm 1, \ldots, \pm 5$.

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