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A (p, q)-extension of Srivastava's triple hypergeometric function H_B and its properties

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Abstract

In this paper, we obtain a (p, q)-extension of Srivastava's triple hypergeometric function $H_B(\cdot)$, by using the extended Beta function $B_{p,q}(x, y)$ introduced by Choi *et al.* [Honam Math. J., **36** (2011) 357–385]. We give some of the main properties of this extended function, which include several integral representations involving Exton's hypergeometric function, the Mellin transform, a differential formula, recursion formulas and a bounded inequality. In addition, a new integral representation of the extended Srivastava triple hypergeometric function involving Laguerre polynomials is obtained.

MSC: 33C60; 33C70; 33C65; 33B15; 33C05; 33C45

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1. Introduction and Preliminaries

In the present paper, we employ the following notations:

$$\mathbf{N} := \{1, 2, ...\}, \quad \mathbf{N}_0 := \mathbf{N} \cup \{0\}, \quad \mathbf{Z}_0^- := \mathbf{Z}^- \cup \{0\},$$

where the symbols $\mathbf{N}$ and $\mathbf{Z}$ denote the set of integer and natural numbers; as usual, the symbols $\mathbf{R}$ and $\mathbf{C}$ denote the set of real and complex numbers, respectively.

In the available literature, the hypergeometric series and its generalizations appear in various branches of mathematics associated with applications. A large number of triple hypergeometric functions have been introduced and investigated. The work of Srivastava and Karlsson [23, Chapter 3] provides a table of 205 distinct triple hypergeometric functions.

Srivastava introduced the triple hypergeometric functions H_A, H_B and H_C of the second order in [20, 21]. It is known that H_C and H_B are generalizations of Appell's hypergeometric functions F_1 and F_2, while H_A is the generalization of both F_1 and F_2.

In the present study, we confine our attention to Srivastava's triple hypergeometric function H_B given by [23, p. 43, 1.5(11) to 1.5(13)] (see also [20] and [22, p. 68])

$$H_B(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{m+k}(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}$$

$$= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B(b_1+m+k, b_2+m+n)}{B(b_1, b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}. \tag{1.1}$$

Here $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0,\ \lambda \in \mathbf{C}\backslash\{0\}) \\ \lambda(\lambda + 1)...(\lambda + n - 1), & (n \in \mathbf{N},\ \lambda \in \mathbf{C}) \end{cases}$$

and $B(\alpha, \beta)$ denotes the classical Beta function defined by [15, (5.12.1)]

$$B(\alpha, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, & (\Re(\alpha) > 0, \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & ((\alpha, \beta) \in \mathbf{C}\backslash\mathbf{Z}_0^-). \end{cases} \tag{1.2}$$

The convergence region for Srivastava's triple hypergeometric series $H_B(\cdot)$ is given in [12, p.243] as $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where α, β, γ satisfy the relation $\alpha+\beta+\gamma+2\sqrt{\alpha\beta\gamma} = 1$.

A different type of triple hypergeometric function is Exton's function $X_4(\cdot)$, which is defined by (see [11] and [23, p. 84, Entry (45a)])

$$X_4(b_1, b_2; c_1, c_2, c_3; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{2m+n+k}(b_2)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}. \tag{1.3}$$

The convergence region for this series is $2\sqrt{|x|} + (\sqrt{|y|} + \sqrt{|z|})^2 < 1$. We shall also find it convenient to introduce two additional parameters r, s into $H_B(\cdot)$ in the form

$$H_B^{(r,s)}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) :=$$

$$\sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B(b_1+r+m+k, b_2+s+m+n)}{B(b_1, b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}, \tag{1.4}$$

which reduces to (1.1) when $r = s = 0$.

Extension and generalizations of hypergeometric series have appeared in the following research papers [1, 4, 9, 10, 13, 17]. In 1997, Chaudhry *et al.* [2, Eq. (1.7)] gave a p-extension of the Beta function $B(x, y)$ given by

$$B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \qquad (\Re(p) > 0) \tag{1.5}$$

and they proved that this extension has connections with the Macdonald and Whittaker functions, and the error function. Also, Chaudhry *et al.* [3] extended the Gauss hypergeometric series ${}_2F_1(\cdot)$ and its integral representations. Recently, Choi *et al.* [9] have given a further extension of the extended Beta function $B(x,y;p)$ by adding one more parameter q, which we denote and define by

$$B_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt, \tag{1.6}$$

where $\Re(p) > 0$, $\Re(q) > 0$. When $p = q$ this function reduces to $B(x,y;p)$. Also, Choi *et al.* [9] studied an extension of the Gauss hypergeometric function and its integral representation based on the extended Beta function (1.6). The Appell hypergeometric function $F_1(\cdot)$, defined by

$$F_1(b_1,b_2,b_3;c_1;x,y) = \sum_{n,m=0}^{\infty} \frac{(b_2)_m(b_3)_n B(b_1+m+n,c_1-b_1)}{B(b_1,c_1-b_1)} \frac{x^m}{m!}\frac{y^n}{n!} \quad (|x|<1,\ |y|<1),$$

has been extended by replacing the numerator Beta function (of the same arguments) with that in (1.5) by Özarslan and Özergin [16] and with that in (1.6) by Parmar and Pogány [18].

Motivated by some of the above-mentioned extensions of special functions, many authors have studied integral representations of the $H_B(\cdot)$ function; see [5, 6, 7, 8]. Our aim in this paper is to introduce a (p,q)-extension of Srivastava's triple hypergeometric function $H_B(\cdot)$ in (1.1), which we denote by $H_{B,p,q}(\cdot)$, based on the extended Beta function in (1.6). We then systematically investigate some properties of this extended function, namely the Mellin transform, a differential formula, recursion formulas and a bounded inequality satisfied by this function. It is hoped that this extension will find application in various branches of applied mathematics and mathematical physics. Similar extensions of the other Srivastava triple hypergeometric functions are under investigation.

The plan of this paper as follows. The extended Srivastava hypergeometric function $H_{B,p,q}(\cdot)$ is defined in Section 2 and some integral representations are presented involving Exton's function X_4 and the Laguerre polynomials. The main properties of $H_{B,p,q}(\cdot)$ namely, its Mellin transform, a differential formula, a bounded inequality and recursion formulas are established in Sections 3-6. Some concluding remarks are made in Section 7.

2. The $(p,q)-$extended Srivastava triple hypergeometric function $H_{B,p,q}(\cdot)$

Srivastava introduced the triple hypergeometric function $H_B(\cdot)$, together with its integral representations, in [20] and [22]. Here we consider the following (p,q)-extension of this function, which we denote by $H_{B,p,q}(\cdot)$, based on the extended Beta function $B_{p,q}(x,y)$ defined in (1.6). This is given by

$$\begin{aligned} &H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z) \\ &= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}, \end{aligned} \tag{2.1}$$

where the parameters $b_1, b_2, b_3 \in \mathbf{C}$ and $c_1, c_2, c_2 \in \mathbf{C}\backslash\mathbf{Z}_0^-$. The region of convergence is $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where $\alpha + \beta + \gamma + 2\sqrt{\alpha\beta\gamma} = 1$. This definition clearly reduces to the original classical function in (1.1) when $p = q = 0$.

Several integral representations for $H_{B,p,q}(\cdot)$ involving Exton's triple hypergeometric function in (1.3) can be given. We have

Theorem 1. *Each of the following integral representations of the extended Srivastava triple hypergeometric function $H_{B,p,q}(\cdot)$ holds for $\Re(p) > 0$, $\Re(q) > 0$ and $\min\{\Re(b_1), \Re(b_2)\} > 0$:*

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} \int_0^1 t^{b_1-1}(1-t)^{b_2-1} \exp\Big(-\frac{p}{t} - \frac{q}{1-t}\Big)$$
$$\times X_4(b_1 + b_2, b_3; c_1, c_2, c_3; xt(1-t), y(1-t), zt)\, dt; \tag{2.2}$$

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \frac{(\beta-\gamma)^{b_1}(\alpha-\gamma)^{b_2}}{(\beta-\alpha)^{b_1+b_2-1}} \frac{\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)}$$
$$\times \int_\alpha^\beta \frac{(\xi-\alpha)^{b_1-1}(\beta-\xi)^{b_2-1}}{(\xi-\gamma)^{b_1+b_2}} \exp\Big(-\frac{p}{\sigma_2} - \frac{q}{\sigma_1}\Big) X_4(b_1 + b_2, b_3; c_1, c_2, c_3; \sigma_1\sigma_2 x, \sigma_1 y, \sigma_2 z)\, d\xi, \tag{2.3}$$

where α, β, γ are real parameters satisfying $\gamma < \alpha < \beta$ and

$$\sigma_1 = \frac{(\alpha-\gamma)(\beta-\xi)}{(\beta-\alpha)(\xi-\gamma)}, \qquad \sigma_2 = \frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)};$$

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \frac{2\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} \int_0^{\frac{\pi}{2}} (\sin^2\xi)^{b_1-\frac12}(\cos^2\xi)^{b_2-\frac12} \exp\Big(-\frac{p}{\sigma_2} - \frac{q}{\sigma_1}\Big)$$
$$\times X_4(b_1 + b_2, b_3; c_1, c_2, c_3; \sigma_1\sigma_2 x, \sigma_1 y, \sigma_2 z)\, d\xi, \tag{2.4}$$

where

$$\sigma_1 = \cos^2\xi, \qquad \sigma_2 = \sin^2\xi;$$

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \frac{2(1+\lambda)^{b_1}\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} \int_0^{\frac{\pi}{2}} \frac{(\sin^2\xi)^{b_1-\frac12}(\cos^2\xi)^{b_2-\frac12}}{(1+\lambda\sin^2\xi)^{b_1+b_2}}$$
$$\times \exp\Big(-\frac{p}{\sigma_2} - \frac{q}{\sigma_1}\Big) X_4(b_1 + b_2, b_3; c_1, c_2, c_3; \sigma_1\sigma_2 x, \sigma_1 y, \sigma_2 z)\, d\xi, \tag{2.5}$$

where

$$\sigma_1 = \frac{\cos^2\xi}{1+\lambda\sin^2\xi}, \qquad \sigma_2 = \frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi} \qquad (\lambda > -1);$$

and

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \frac{2\lambda^{b_1}\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} \int_0^{\frac{\pi}{2}} \frac{(\sin^2\xi)^{b_1-\frac12}(\cos^2\xi)^{b_2-\frac12}}{(\cos^2\xi+\lambda\sin^2\xi)^{b_1+b_2}}$$
$$\times \exp\Big(-\frac{p}{\sigma_2} - \frac{q}{\sigma_1}\Big) X_4(b_1 + b_2, b_3; c_1, c_2, c_3; \sigma_1\sigma_2 x, \sigma_1 y, \sigma_2 z)\, d\xi, \tag{2.6}$$

where

$$\sigma_1 = \frac{\cos^2\xi}{\cos^2\xi+\lambda\sin^2\xi}, \qquad \sigma_2 = \frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi} \qquad (\lambda > 0).$$

Proof: The proof of the first integral representation (2.2) follows by use of the extended beta function (1.6) in (2.1), a change in the order of integration and summation (by uniform convergence of the integral) and, after simplification, use of Exton's triple hypergeometric function (1.3), to obtain the right-hand side of the result (2.2). The integral representations (2.3)-(2.6) can be proved directly by using the following transformations

$$\begin{aligned}
(2.3): \qquad t &= \frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)}, \quad \frac{dt}{d\xi} = \frac{(\beta-\gamma)(\alpha-\gamma)}{(\beta-\alpha)(\xi-\gamma)^2}, \\
(2.4): \qquad t &= \sin^2\xi, \quad \frac{dt}{d\xi} = 2\sin\xi\cos\xi \\
(2.5): \qquad t &= \frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi}, \quad \frac{dt}{d\xi} = \frac{2(1+\lambda)\sin\xi\cos\xi}{(1+\lambda\sin^2\xi)^2}, \\
(2.6): \qquad t &= \frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi}, \quad \frac{dt}{d\xi} = \frac{2\lambda\sin\xi\cos\xi}{(\cos^2\xi+\lambda\sin^2\xi)^2}
\end{aligned}$$

in turn in (2.2) to obtain the right-hand side of each result.

Theorem 2. *The following representation of $H_{B,p,q}(\cdot)$ associated with Laguerre polynomials holds true:*

$$H_{B,p,q}(b_1, b_2, b_3 : c_1, c_2, c_3; x, y, z) = \frac{e^{-p-q}\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)} \sum_{n,m=0}^{\infty} L_n(p)L_m(q) \int_0^1 t^{b_1+m}(1-t)^{b_2+n}$$

$$\times X_4(b_1+b_2, b_3; c_1, c_2, c_3; xt(1-t), y(1-t), zt)\, dt, \tag{2.7}$$

where $p > 0$, $q > 0$ *and* $\min\{\Re(b_1), \Re(b_2)\} > 0$.

Proof: A representation of exponential factor in (1.6) can be obtained in terms of Laguerre polynomials. The definition of the Laguerre polynomials $L_m(x)$ $(m \in \mathbf{N}_0)$ is given by the generating function [19, p. 202]

$$\exp\left(-\frac{xt}{1-t}\right) = (1-t)\sum_{n=0}^{\infty} t^n L_n(x) \qquad (-1 < t < 1)$$

for $x > 0$. From this we see that

$$\exp\left(-\frac{q}{1-t}\right) = e^{-q}(1-t)\sum_{m=0}^{\infty} t^m L_m(q) \qquad (-1 < t < 1)$$

and, upon replacement of t by $1-t$,

$$\exp\left(-\frac{p}{t}\right) = e^{-p}\, t \sum_{n=0}^{\infty} (1-t)^n L_n(p) \qquad (0 < t < 2).$$

Combining these last two expressions, we obtain

$$\exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) = e^{-p-q} \sum_{n,m=0}^{\infty} t^{m+1}(1-t)^{n+1} L_n(p)L_m(q) \qquad (0 < t < 1). \tag{2.8}$$

Applying (2.8) in (2.2), we then obtain the required result.

3. The Mellin transform for $H_{B,p,q}(\cdot)$

The Mellin transform of a locally integrable function $f(x,y)$ with indices r and s given in [14, p.193, Sec.(2.1), Entry (1.1)] is defined by

$$\Phi(r,s) \equiv \mathcal{M}\{f(x,y)\}(r,s) = \int_0^\infty\int_0^\infty x^{r-1}y^{s-1}f(x,y)\,dxdy \tag{3.1}$$

which defines an analytic function in the strips of analyticity $\alpha < \Re(r) < \beta$, $\gamma < \Re(s) < \delta$. The inverse Mellin transform is defined by

$$f(x,y) = \mathcal{M}^{-1}\{\Phi(r,s)\} = \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty} x^{-r}y^{-s}\Phi(r,s)\,ds\,dr,$$

where $\alpha < c < \beta$, $\gamma < d < \delta$.

Theorem 3. *The following Mellin transform of the extended Srivastava triple hypergeometric function $H_{B,p,q}(\cdot)$ holds true:*

$$\begin{aligned}
&\mathcal{M}\left\{H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\right\}(r,s)\\
&\qquad= \int_0^\infty\int_0^\infty p^{r-1}q^{s-1}H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\,dp\,dq,\\
&\qquad= \Gamma(r)\Gamma(s)\,H_B^{(r,s)}\left(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right),
\end{aligned} \tag{3.2}$$

where $H_B^{(r,s)}$ is defined in (1.4) and $\Re(p) > 0$, $\Re(q) > 0$, $\Re(r) > 0$, $\Re(s) > 0$, $\Re(b_1 + r) > 0$, $\Re(b_2 + s) > 0$ and $c_1, c_2, c_3 \in \mathbf{C}\backslash\mathbf{Z}_0^-$.

Proof: Substituting the extended Srivastava function (2.1) into the double integral in (3.2) and changing the order of integration (by the uniform convergence of the integral), we obtain

$$\begin{aligned}
\mathcal{M}\left\{H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\right\}(r,s) &= \frac{1}{B(b_1,b_2)}\sum_{m,n,k\geq 0}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\\
\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}B_{p,q}(b_1+m+k,b_2+m+n)\,dp\,dq.
\end{aligned} \tag{3.3}$$

Applying the following integral formula [9, Eq. (2.1)]

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}B_{p,q}(x,y)\,dp\,dq = \Gamma(r)\Gamma(s)B(x+r,y+s),$$

where $\Re(p) > 0$, $\Re(q) > 0$, $\Re(r) > 0$, $\Re(s) > 0$, $\Re(x + r) > 0$, $\Re(y + s) > 0$ in (3.3), we obtain

$$\begin{aligned}
&\mathcal{M}\left\{H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\right\}(r,s)\\
&= \Gamma(r)\Gamma(s)\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\frac{B(b_1+r+m+k,b_2+s+m+n)}{B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.
\end{aligned}$$

Finally, in view of the definition in (1.4), we obtain right-hand side of the Mellin transform stated in (3.2).

Corollary 1: The following inverse Mellin formula for $H_{B,p,q}(\cdot)$ holds:

$$H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$
$$=\frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty} p^{-r}q^{-s}\Gamma(r)\Gamma(s)H_B^{(r,s)}\left(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right)\,ds\,dr \tag{3.4}$$

where $c>0$, $d>0$.

4. A differentiation formula for $H_{B,p,q}(\cdot)$

Theorem 4. *The following derivative formula for $H_{B,p,q}(\cdot)$ holds:*

$$\frac{\partial^{M+N+K}}{\partial x^M\partial y^N\partial z^K}\,H_{B,p,q}\left(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right)=\frac{(b_1)_{M+K}(b_2)_{M+N}(b_3)_{N+K}}{(c_1)_M(c_2)_N(c_3)_K}$$
$$\times H_{B,p,q}(b_1{+}M{+}K,b_2{+}M{+}N,b_3{+}N{+}K;c_1{+}M,c_2{+}N,c_3{+}K;x,y,z), \tag{4.1}$$

where $M,\ N,\ K\in\mathbf{N}_0$.

Proof: If we differentiate partially the series for $\mathcal{H}\equiv H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$ in (2.1) with respect to x we obtain

$$\frac{\partial\mathcal{H}}{\partial x}=\sum_{m=1}^{\infty}\sum_{n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,q}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\,\frac{x^{m-1}}{(m-1)!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Making use of the fact that

$$B(b_1,b_2)=\frac{(b_1+b_2)_2}{b_1b_2}\,B(b_1+1,b_2+1) \tag{4.2}$$

and $(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n$, we have upon setting $m\to m+1$

$$\frac{\partial\mathcal{H}}{\partial x}=\frac{b_1b_2}{c_1}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2+2)_{2m+n+k}(b_3)_{n+k}}{(c_1+1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,q}(b_1{+}1{+}m{+}k,b_2{+}1{+}m{+}n)}{B(b_1+1,b_2+1)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$=\frac{b_1b_2}{c_1}\,H_{B,p,q}(b_1+1,b_2+1,b_3;c_1+1,c_2,c_3;x,y,z). \tag{4.3}$$

Repeated application of (4.3) then yields for $M=1,2,\ldots$

$$\frac{\partial^M\mathcal{H}}{\partial x^M}=\frac{(b_1)_M(b_2)_M}{(c_1)_M}\,H_{B,p,q}(b_1+M,b_2+M,b_3;c_1+M,c_2,c_3;x,y,z).$$

A similar reasoning shows that

$$\frac{\partial^{M+1}\mathcal{H}}{\partial x^M\partial y}=\frac{(b_1)_M(b_2)_M}{(c_1)_M}\sum_{m,k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(b_1{+}b_2{+}2M)_{2m+n+k}(b_3)_{n+k}}{(c_1{+}M)_m(c_2)_n(c_3)_k}$$
$$\times\frac{B_{p,\nu}(b_1{+}M{+}m{+}k,b_2{+}M{+}n{+}k)}{B(b_1{+}M,b_2{+}M)}\,\frac{x^m}{m!}\,\frac{y^{n-1}}{(n-1)!}\,\frac{z^k}{k!}$$

$$= \frac{(b_1)_M (b_2)_{M+1} b_3}{(c_1)_M c_2} H_{B,p,q}(b_1+M, b_2+M+1, b_3+1; c_1+M, c_2+1, c_3; x, y, z) \tag{4.4}$$

upon putting $n \to n+1$ and using the property of the Beta function in (1.2). Repeated differentiation of (4.4) N times with respect to y then produces

$$\frac{\partial^{M+N}\mathcal{H}}{\partial x^M \partial y^N} = \frac{(b_1)_M (b_2)_{M+N} (b_3)_N}{(c_1)_M (c_2)_N} H_{B,p,q}(b_1+M, b_2+M+N, b_3+N; c_1+M, c_2+N, c_3; x, y, z).$$

Application of the same procedure to deal with differentiation with respect to z then yields the result stated in (4.1).

5. An upper bound for $H_{B,p,q}(\cdot)$

Theorem 5. *Let the parameters b_j, c_j $(1 \le j \le 3)$ be positive and the variables $x, y, z \in \mathbf{C}$. Further suppose that $\Re(p) > 0$ and $\Re(q) > 0$. Then the following bounded inequality for $H_{B,p,q}(\cdot)$ holds:*

$$|H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)| < \lambda_E \, H_B(b_1, b_2, b_3; c_1, c_2, c_3; |x|, |y|, |z|), \tag{5.1}$$

where $\lambda_E = \exp[-\Re(p) - \Re(q) - 2\sqrt{\Re(p)\Re(q)}\,]$.

Proof: We shall assume that the parameters b_j, $c_j > 0$ $(1 \le j \le 3)$ and that $\Re(p) > 0$, $\Re(q) > 0$ with $x, y, z \in \mathbf{C}$. Then from (2.1) we have

$$|H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)|$$
$$\le \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k} (b_3)_{n+k}}{(c_1)_m (c_2)_n (c_3)_k} \frac{|B_{p,q}(b_1+m+k, b_2+m+n)|}{B(b_1, b_2)} \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!}. \tag{5.2}$$

Now from the definition of the extended Beta function $B_{p,q}(a, b)$ in (1.6), with $a, b > 0$, we have

$$|B_{p,q}(a, b)| \le \int_0^1 t^{a-1}(1-t)^{b-1} |E_{p,q}(t)| \, dt, \qquad E_{p,q}(t) := \exp\Big(-\frac{p}{t} - \frac{q}{1-t}\Big)$$
$$< \int_0^1 t^{a-1}(1-t)^{b-1} E_{\Re(p),\Re(q)}(t) \, dt.$$

From the fact that $E_{\Re(p),\Re(q)}(t)$ attains its maximum value at $t^* = r/(1+r)$, with $r = \sqrt{\Re(p)/\Re(q)}$, we then deduce that

$$|B_{p,q}(a, b)| < \lambda_E \, B(a, b), \qquad \lambda_E := \exp[-\Re(p) - \Re(q) - 2\sqrt{\Re(p)\Re(q)}\,].$$

Consequently, from (5.2), we obtain

$$|H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)|$$
$$< \lambda_E \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k} (b_3)_{n+k}}{(c_1)_m (c_2)_n (c_3)_k} \frac{B(b_1+m+k, b_2+m+n)}{B(b_1, b_2)} \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!}.$$

Identification of this last sum by means of (1.1) then yields the result stated in (5.1).

We remark that if $p = q = \ell > 0$ then $\lambda_E = e^{-4\ell}$.

6. Recursion formulas for $H_{B,p,q}(\cdot)$

In this section, we obtain two recursion formulas for the extended Srivastava function $H_{B,p,q}(\cdot)$. The first formula gives a recursion with respect to the numerator parameter b_3, and the second a recursion with respect to any one of the denominator parameters c_j $(1 \le j \le 3)$.

Theorem 6. *The following recursion for $H_{B,p,q}(\cdot)$ with respect to the numerator parameter b_3 holds:*

$$H_{B,p,q}(b_1,b_2,b_3+1;c_1,c_2,c_3;x,y,z) = H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$

$$+\frac{yb_2}{c_2}H_{B,p,q}(b_1,b_2+1,b_3+1;c_1,c_2+1,c_3;x,y,z)+\frac{zb_1}{c_3}H_{B,p,q}(b_1+1,b_2,b_3+1;c_1,c_2,c_3+1;x,y,z). \tag{6.1}$$

Proof. From (2.1) and the result $(b_3+1)_{n+k} = (b_3)_{n+k}(1+n/b_3+k/b_3)$, we obtain

$$\begin{aligned}
&H_{B,p,q}(b_1,b_2,b_3+1;c_1,c_2,c_3;x,y,z)\\
&= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3+1)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\frac{B_{p,q}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\\
&= H_{B,p,q}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\\
&+\frac{y}{b_3}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\frac{B_{p,q}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\frac{x^m}{m!}\frac{y^{n-1}}{(n-1)!}\frac{z^k}{k!}\\
&+\frac{z}{b_3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\frac{B_{p,q}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^{k-1}}{(k-1)!}.
\end{aligned} \tag{6.2}$$

Consider the first sum in (6.2) which we denote by S. Put $n \to n+1$ and use the identity $(a)_{n+1} = a(a+1)_n$ to find

$$S = \frac{y}{b_3}\sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+1+k}(b_3)_{n+1+k}}{(c_1)_m(c_2)_{n+1}(c_3)_k}\frac{B_{p,q}(b_1+m+k,b_2+1+m+n)}{B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$= \frac{y(b_1+b_2)}{c_2}\sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2+1)_{2m+n+k}(b_3+1)_{n+k}}{(c_1)_m(c_2+1)_n(c_3)_k}\frac{B_{p,q}(b_1+m+k,b_2+1+m+n)}{B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Using the fact that

$$B(b_1,b_2) = \frac{b_1+b_2}{b_2}B(b_1,b_2+1),$$

we then obtain

$$\begin{aligned}
S &= \frac{yb_2}{c_2}\sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2+1)_{2m+n+k}(b_3+1)_{n+k}}{(c_1)_m(c_2+1)_n(c_3)_k}\frac{B_{p,q}(b_1+m+k,b_2+1+m+n)}{B(b_1,b_2+1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\\
&= \frac{yb_2}{c_2}H_{B,p,q}(b_1,b_2+1,b_3+1;c_1,c_2+1,c_3;x,y,z).
\end{aligned} \tag{6.3}$$

Proceeding in a similar manner for the second series in (6.2) with $k \to k+1$, we find that this sum can be expressed as

$$\frac{zb_1}{c_3} H_{B,p,q}(b_1+1, b_2, b_3+1; c_1, c_2, c_3+1; x, y, z). \tag{6.4}$$

Combination of (6.3) and (6.4) with (6.2) then produces the result stated in (6.1).

Corollary 2: From (6.1) the following recursion holds

$$H_{B,p,q}(b_1, b_2, b_3+N; c_1, c_2, c_3; x, y, z) = H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)$$
$$+\frac{yb_2}{c_2}\sum_{\ell=1}^{N} H_{B,p,q}(b_1, b_2+1, b_3+\ell; c_1, c_2+1, c_3; x, y, z)$$
$$+\frac{zb_1}{c_3}\sum_{\ell=1}^{N} H_{B,p,q}(b_1+1, b_2, b_3+\ell; c_1, c_2, c_3+1; x, y, z) \tag{6.5}$$

for positive integer N.

Theorem 7. *The following 3-term recursion for $H_{B,p,q}(\cdot)$ with respect to the denominator parameter c_1 holds:*

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) =$$
$$H_{B,p,q}(b_1, b_2, b_3; c_1+1, c_2, c_3; x, y, z) + \frac{xb_1b_2}{c_1(c_1+1)} H_{B,p,q}(b_1+1, b_2+1, b_3; c_1+2, c_2, c_3; x, y, z). \tag{6.6}$$

Permutation of the c_j enables analogous recursions in the denominator parameters c_2 and c_3 to be obtained.

Proof. Consider the case when c_1 is reduced by 1, namely

$$H \equiv H_{B,p,q}(b_1, b_2, b_3; c_1-1, c_2, c_3; x, y, z)$$

and use $(c_1-1)_m = (c_1)_m/\{1+\frac{m}{c_1-1}\}$. Then

$$H = \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1-1)_m(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+m+k, b_2+m+n)}{B(b_1,b_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+m+k, b_2+m+n)}{B(b_1,b_2)} \left(1+\frac{m}{c_1-1}\right)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$= H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)$$
$$+\frac{x}{c_1-1}\sum_{m=1}^{\infty}\sum_{n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+m+k, b_2+m+n)}{B(b_1,b_2)} \frac{x^{m-1}}{(m-1)!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Putting $m \to m+1$ in the above sum, we obtain

$$\frac{x}{c_1-1}\sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+2+n+k}(b_3)_{n+k}}{(c_1)_{m+1}(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+1+m+k, b_2+1+m+n)}{B(b_1,b_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$= \frac{x(b_1+b_2)_2}{c_1(c_1-1)} \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2+2)_{2m+n+k}(b_3)_{n+k}}{(c_1+1)_m(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+1+m+k, b_2+1+m+n)}{B(b_1,b_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Using (4.2), we find that this last sum becomes

$$\frac{xb_1b_2}{c_1(c_1-1)} \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2+2)_{2m+n+k}(b_3)_{n+k}}{(c_1+1)_m(c_2)_n(c_3)_k} \frac{B_{p,q}(b_1+1+m+k, b_2+1+m+n)}{B(b_1+1,b_2+1)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$= \frac{xb_1b_2}{c_1(c_1-1)} H_{B,p,q}(b_1+1, b_2+1, b_3; c_1+1, c_2, c_3; x, y, z).$$

This then yields the recurrence relation (in c_1) given by

$$H_{B,p,q}(b_1, b_2, b_3; c_1-1, c_2, c_3; x, y, z) =$$

$$H_{B,p,q}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) + \frac{xb_1b_2}{c_1(c_1-1)} H_{B,p,q}(b_1+1, b_2+1, b_3; c_1+1, c_2, c_3; x, y, z).$$

Replacement of c_1 by c_1+1 then yields the result stated in (6.6).

7. Concluding remarks

In this paper, we have introduced the (p,q)-extended Srivastava triple hypergeometric function given by $H_{B,p,q}(\cdot)$ in (2.1). We have given some integral representations of this function that involve Exton's triple hypergeometric function X_4. We have also established some properties of the function $H_{B,p,q}(\cdot)$, namely the Mellin transform, a differential formula, a bounded inequality and some recursion relations. In addition, we have given an integral representation involving Laguerre polynomials.

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