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INEQUALITIES AND ASYMPTOTIC EXPANSIONS RELATED TO THE VOLUME OF THE UNIT BALL IN \mathbb{R}^n

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ABSTRACT. Let $\Omega_n = \pi^{n/2}/\Gamma(\frac{n}{2} + 1)$ ($n \in \mathbb{N}$) denote the volume of the unit ball in \mathbb{R}^n . In this paper, we present asymptotic expansions and inequalities related to Ω_n and the quantities:

$$\frac{\Omega_{n-1}}{\Omega_n}, \quad \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} \quad \text{and} \quad \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}}.$$

1. INTRODUCTION

In the recent past, several researchers have established interesting properties of the volume Ω_n of the unit ball in \mathbb{R}^n ,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \quad n \in \mathbb{N} := \{1, 2, \dots\},$$

including monotonicity properties, inequalities and asymptotic expansions.

Böhm and Hertel [7, p. 264] pointed out that the sequence $\{\Omega_n\}_{n \geq 1}$ is not monotonic for $n \geq 1$. Indeed, we have

$$\Omega_n < \Omega_{n+1} \quad \text{if} \quad 1 \leq n \leq 4 \quad \text{and} \quad \Omega_n > \Omega_{n+1} \quad \text{if} \quad n \geq 5.$$

Anderson *et al.* [5] showed that $\{\Omega_n^{1/n}\}_{n \geq 1}$ is monotonically decreasing to zero, while Anderson and Qiu [4] proved that the sequence $\{\Omega_n^{1/(n \ln n)}\}_{n \geq 2}$ decreases to $e^{-1/2}$. Guo and Qi [14] proved that the sequence $\{\Omega_n^{1/(n \ln n)}\}_{n \geq 2}$ is logarithmically convex. Klain and Rota [16] proved that the sequence $\{n\Omega_n/\Omega_{n-1}\}_{n \geq 1}$ is increasing.

Diverse sharp inequalities for the volume of the unit ball in \mathbb{R}^n have been established [2, 3, 6, 8, 11, 19, 21–24, 28]. For example, Alzer [2] proved that for $n \in \mathbb{N}$,

$$a_1 \Omega_{n+1}^{n/(n+1)} \leq \Omega_n < b_1 \Omega_{n+1}^{n/(n+1)}, \quad (1.1)$$

$$\sqrt{\frac{n+a_2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b_2}{2\pi}}, \quad (1.2)$$

$$\left(1 + \frac{1}{n}\right)^{a_3} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{b_3}, \quad (1.3)$$

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with the best possible constants

$$\begin{aligned} a_1 &= \frac{2}{\sqrt{\pi}} = 1.1283\dots, & b_1 &= \sqrt{e} = 1.6487\dots, \\ a_2 &= \frac{1}{2}, & b_2 &= \frac{\pi}{2} - 1 = 0.5707\dots, \\ a_3 &= 2 - \frac{\ln \pi}{\ln 2} = 0.3485\dots, & b_3 &= \frac{1}{2}. \end{aligned}$$

The double inequality (1.2) refines a result due to Borgwardt [8, p. 253], who proved (1.2) with $a_2 = 0$ and $b_2 = 1$. Mortici [22, Theorem 3] obtained the following inequality:

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi n}}, \quad n \in \mathbb{N}, \quad (1.4)$$

which improves the right-hand side of (1.2). Ban and Chen [6, Theorem 3.1] improved (1.4) and obtained the following double inequality:

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi(n + \vartheta_1)}} \leq \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi(n + \vartheta_2)}}, \quad n \in \mathbb{N}, \quad (1.5)$$

with best possible constants

$$\vartheta_1 = \frac{13 - 4\pi}{4\pi - 12} = 0.7656283\dots \quad \text{and} \quad \vartheta_2 = \frac{1}{2}.$$

Merkle [21] improved the left-hand side of (1.3) and obtained the following result:

$$\left(1 + \frac{1}{n+1}\right)^{1/2} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}, \quad n \in \mathbb{N}. \quad (1.6)$$

Chen and Lin [11, Theorem 3.1] developed (1.6) to produce the following symmetric double inequality:

$$\left(1 + \frac{1}{n+1}\right)^\alpha < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leq \left(1 + \frac{1}{n+1}\right)^\beta, \quad n \in \mathbb{N}, \quad (1.7)$$

with the best possible constants

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2\ln 2 - \ln \pi}{\ln 3 - \ln 2} = 0.5957713\dots,$$

Ban and Chen [6, Theorem 3.2] proved, for $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n + \theta_1}\right)^{1/2} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n + \theta_2}\right)^{1/2}, \quad (1.8)$$

with best possible constants

$$\theta_1 = \frac{2\pi^2 - 16}{16 - \pi^2} = 0.60994576\dots \quad \text{and} \quad \theta_2 = \frac{1}{2}.$$

Alzer [3] continued the work on this subject and offered new inequalities. For example, Alzer [3, Theorem 4] proved that for $n \geq 2$,

$$\frac{\alpha^*}{\sqrt{n}} \leq \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \frac{\beta^*}{\sqrt{n}}, \quad (1.9)$$

with best possible constants

$$\alpha^* = \frac{3\sqrt{2}\pi}{6+4\pi} = 0.7178\dots \quad \text{and} \quad \beta^* = \sqrt{2\pi} = 2.5066\dots,$$

while Chen and Lin [11, Theorem 3.3] proved that for $n \in \mathbb{N}$,

$$\sqrt{\frac{2\pi}{n+a^*}} \leq \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \sqrt{\frac{2\pi}{n+b^*}}, \quad (1.10)$$

with best possible constants

$$a^* = \frac{\pi(1+\pi)^2}{2} - 1 = 25.94353\dots, \quad b^* = \frac{1}{2} + 4\pi = 13.06637\dots$$

Chen and Lin [11, Theorem 3.4] showed that for $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{\pi(n+\theta)}} \left(\frac{2\pi e}{n}\right)^{n/2} \leq \Omega_n < \frac{1}{\sqrt{\pi(n+\vartheta)}} \left(\frac{2\pi e}{n}\right)^{n/2}, \quad (1.11)$$

with best possible constants

$$\theta = \frac{e}{2} - 1 = 0.3591409\dots, \quad \vartheta = \frac{1}{3}.$$

Recently, Mortici [24] constructed asymptotic series associated with some expressions involving the volume of the n -dimensional unit ball. New refinements and improvements of some old and recent inequalities for Ω_n are also presented. Lu and Zhang [19] established a general continued fraction approximation for the n th root of the volume of the unit n -dimensional ball, and then obtained related inequalities.

In this paper, we present asymptotic expansions and inequalities related to Ω_n and the quantities:

$$\frac{\Omega_{n-1}}{\Omega_n}, \quad \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} \quad \text{and} \quad \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}}.$$

The numerical values given in this paper have been calculated via the computer program MAPLE 17.

2. LEMMAS

The following lemmas are required in our present investigation.

Lemma 2.1 ([10, Theorem 5]). *Let*

$$A(x) = \sum_{n=1}^{\infty} a_n x^{-n}, \quad x \rightarrow \infty$$

be an asymptotical expansion. Then the composition $\exp(A(x))$ has the asymptotic expansion given by

$$\exp(A(x)) = \sum_{n=0}^{\infty} b_n x^{-n}, \quad x \rightarrow \infty,$$

where

$$b_0 = 1, \quad b_n = \frac{1}{n} \sum_{k=1}^n k a_k b_{n-k}, \quad n \geq 1.$$

Lemma 2.2 ([1, Theorem 8]). *For every $m \in \mathbb{N}_0$, the function*

$$R_m(x) = (-1)^m \left[\ln \Gamma(x+1) - \left(x + \frac{1}{2}\right) \ln x + x - \ln \sqrt{2\pi} - \sum_{j=1}^m \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right] \quad (2.1)$$

is completely monotonic on $(0, \infty)$, where here and elsewhere in this paper an empty sum is understood to be zero. Here B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (2.2)$$

In 2006, Koumandos [17] presented a simpler proof of complete monotonicity of the functions $R_m(x)$, which was further strengthened by Koumandos and Pedersen in [18, Theorem 2.1].

Recall that a function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for } x \in I \quad \text{and } n \in \mathbb{N}_0. \quad (2.3)$$

Dubourdieu [13, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (2.3); see also [15] for a simpler proof of this result.

The gamma function $\Gamma(x)$ is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi (or digamma) function. It is known that

$$\Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

From the inequality $R_m(x) > 0$ for $x > 0$ and $m \in \mathbb{N}_0$, we obtain that for $x > 0$ and $n \in \mathbb{N}_0$,

$$\sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} < \ln \left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \right) < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \quad (2.4)$$

In particular, the choice $n = 1$ in (2.4) yields

$$x \ln x - x + \frac{1}{12x} - \frac{1}{360x^3} < \ln \Gamma(x+1) - \ln \sqrt{2\pi x} < x \ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}, \quad x > 0. \quad (2.5)$$

From the inequality $R'_m(x) < 0$ for $x > 0$ and $m \in \mathbb{N}_0$, we obtain that for $x > 0$ and $n \in \mathbb{N}_0$,

$$\sum_{j=1}^{2n} \frac{B_{2j}}{2jx^{2j}} < \ln x + \frac{1}{2x} - \psi(x+1) < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2jx^{2j}} \quad (2.6)$$

In particular, the choice $n = 1$ in (2.6) yields

$$\frac{1}{12x^2} - \frac{1}{120x^4} < \ln x + \frac{1}{2x} - \psi(x+1) < \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6}, \quad x > 0. \quad (2.7)$$

Let $\Omega_x = \pi^{x/2}/\Gamma(\frac{x}{2} + 1)$. We find that (2.1) can be written as

$$R_m(x) = (-1)^m \left[\ln \left(\frac{1}{\sqrt{2\pi x}} \left(\frac{\pi e}{x} \right)^x \right) - \ln \Omega_{2x} - \sum_{j=1}^m \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right]. \quad (2.8)$$

From the inequality $R_m(x) > 0$ for $x > 0$ and $m \in \mathbb{N}_0$, we then obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi x}} \left(\frac{\pi e}{x}\right)^x \exp\left(-\sum_{i=1}^{2m+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right) &< \Omega_{2x} \\ &< \frac{1}{\sqrt{2\pi x}} \left(\frac{\pi e}{x}\right)^x \exp\left(-\sum_{i=1}^{2m} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right). \end{aligned} \quad (2.9)$$

Replacement of x by $n/2$ in (2.9) then produces the following double inequality for Ω_n :

$$\begin{aligned} \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\sum_{j=1}^{2m+1} \frac{2^{2j} B_{2j}}{4j(2j-1)n^{2j-1}}\right) &< \Omega_n \\ &< \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\sum_{j=1}^{2m} \frac{2^{2j} B_{2j}}{4j(2j-1)n^{2j-1}}\right) \end{aligned} \quad (2.10)$$

for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

Mortici [23, Theorem 2] has given the following double inequality:

$$\begin{aligned} \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\frac{1}{6n} + \frac{1}{45n^3} - \frac{8}{315n^5} + \frac{8}{105n^7} - \frac{128}{297n^9}\right) &< \Omega_n \\ &< \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\frac{1}{6n} + \frac{1}{45n^3} - \frac{8}{315n^5} + \frac{8}{105n^7}\right), \quad n \in \mathbb{N}, \end{aligned} \quad (2.11)$$

which can be seen to follow from (2.10) when $m = 2$.

Lemma 2.3 (see [12, Corollary 1]). *Let $m \in \mathbb{N}_0$. Then for $x > 0$,*

$$\begin{aligned} \sqrt{x} \exp\left(\sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}}\right) \frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) &< \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \\ &< \sqrt{x} \exp\left(\sum_{j=1}^{2m+1} \left(1 - \frac{1}{2^{2j}}\right) \frac{B_{2j}}{j(2j-1)x^{2j-1}}\right). \end{aligned} \quad (2.12)$$

In particular, the choice $m = 1$ in (2.12) yields

$$\frac{1}{2} \ln x + \frac{1}{8x} - \frac{1}{192x^3} < \ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) < \frac{1}{2} \ln x + \frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5}, \quad x > 0. \quad (2.13)$$

3. ASYMPTOTICS AND INEQUALITIES FOR Ω_n

The asymptotic expansion of Ω_n as $n \rightarrow \infty$ is given by

$$\begin{aligned} \Omega_n &= \frac{2\pi^{n/2}}{n} \frac{1}{\Gamma(\frac{n}{2})} \sim \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} \sum_{k=0}^{\infty} \frac{2^k \gamma_k}{n^k} \\ &= \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} \left\{ 1 - \frac{1}{6n} + \frac{1}{72n^2} + \frac{139}{6480n^3} - \frac{571}{155520n^4} - \frac{163879}{6531840n^5} \right. \\ &\quad \left. + \frac{5246819}{1175731200n^6} + \frac{534703531}{7054387200n^7} - \dots \right\}, \end{aligned} \quad (3.1)$$

which follows from the well-known result

$$\frac{1}{\Gamma(z)} \sim \frac{e^z z^{\frac{1}{2}-z}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \gamma_k z^{-k} \quad (z \rightarrow \infty, \quad |\arg z| < \pi);$$

see, for example, [26, p. 32], [27, p. 70]. The γ_k denote the Stirling coefficients defined by the recurrence (with $d_0 = 1$)

$$\gamma_k = \frac{(-2)^k \Gamma(k + \frac{1}{2})}{\sqrt{\pi}} d_{2k}, \quad d_n = \frac{n+1}{n+2} \left\{ \frac{d_{n-1}}{n} - \sum_{j=1}^{n-1} \frac{d_j d_{n-j}}{j+1} \right\} \quad (n \geq 1).$$

The first few values of γ_k are

$$\begin{aligned} \gamma_0 &= 1, & \gamma_1 &= -\frac{1}{12}, & \gamma_2 &= \frac{1}{288}, & \gamma_3 &= \frac{139}{51840}, & \gamma_4 &= -\frac{571}{2488320}, & \gamma_5 &= -\frac{163879}{209018880}, \\ \gamma_6 &= \frac{5246819}{75246796800}, & \gamma_7 &= \frac{534703531}{902961561600}, & \dots & & & & & & \end{aligned}$$

The expansion (3.1) was also given by Mortici in [24, Theorem 1].

Motivated by (3.1) we establish the following theorem.

Theorem 3.1. *Let*¹

$$V(x) = \frac{\Omega_{2x}}{\frac{1}{\sqrt{2\pi x}} \left(\frac{\pi e}{x}\right)^x}. \quad (3.2)$$

Then, for $x \geq 1/2$, we have

$$\sum_{k=0}^5 \gamma_k x^{-k} < V(x) < \sum_{k=0}^7 \gamma_k x^{-k} \quad (3.3)$$

and

$$V'(x) < \sum_{k=1}^5 (-k) \gamma_k x^{-k-1}. \quad (3.4)$$

Proof. Here we only prove the left-hand side of (3.3) as the proof of the right-hand side is analogous. The inequality (2.9) can be written as

$$\exp\left(-\sum_{j=1}^{2m+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right) < V(x) < \exp\left(-\sum_{j=1}^{2m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right). \quad (3.5)$$

The choice $m = 1$ on the left-hand side of (3.5) then yields,

$$\exp\left(-\frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5}\right) < V(x).$$

In order prove the left-hand side of (3.3), it suffices to show for $x \geq 1/2$ that

$$\sum_{k=0}^5 \gamma_k x^{-k} < \exp\left(-\frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5}\right). \quad (3.6)$$

¹In terms of the scaled gamma function $\Gamma^*(x) = (e^x x^{\frac{1}{2}-x}/\sqrt{2\pi})\Gamma(x)$, the function $V(x) = 1/\Gamma^*(x)$. Consequently, the double inequality (3.3) supplies bounds on $1/\Gamma^*(x)$.

The inequality (3.6) is obtained by considering the function $P(x)$ defined, for $x \geq 1/2$, by

$$P(x) := \ln \sum_{k=0}^5 \gamma_k x^{-k} - \left(-\frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5} \right).$$

Differentiation yields

$$P'(x) = \frac{P_4(x - \frac{1}{2})}{20160x^6 P_5(x - \frac{1}{2})},$$

where

$$\begin{aligned} P_4(x) &= 98855357 + 763346136x + 2472851136x^2 + 3459743616x^3 + 1762931184x^4, \\ P_5(x) &= 5486171 + 57666084x + 236795328x^2 + 488436480x^3 + 505128960x^4 \\ &\quad + 209018880x^5. \end{aligned}$$

Thus, we have $P'(x) > 0$ for $x \geq 1/2$. So, $P(x)$ is strictly increasing for $x \geq 1/2$, and we have

$$P(x) < \lim_{t \rightarrow \infty} P(t) = 0 \quad \text{for } x \geq \frac{1}{2}.$$

Hence, (3.6) holds for $x \geq 1/2$.

Since the function $R_m(x)$ defined by (2.8) is completely monotonic on $(0, \infty)$, we have, for $x > 0$ and $m \in \mathbb{N}_0$,

$$R'_m(x) = (-1)^{m+1} \left[\frac{V'(x)}{V(x)} - \sum_{j=1}^m \frac{B_{2j}}{2jx^{2j}} \right] < 0, \quad \text{so that } V'(x) < V(x) \sum_{j=1}^{2m+1} \frac{B_{2j}}{2jx^{2j}}.$$

Using the right-hand side of (3.3), we obtain that

$$\begin{aligned} V'(x) &< V(x) \sum_{j=1}^3 \frac{B_{2j}}{2jx^{2j}} \\ &< \left(1 - \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3} - \frac{571}{2488320x^4} - \frac{163879}{209018880x^5} \right. \\ &\quad \left. + \frac{5246819}{75246796800x^6} + \frac{534703531}{902961561600x^7} \right) \left(\frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} \right) \\ &= \frac{1}{12x^2} - \frac{1}{144x^3} - \frac{139}{17280x^4} + \frac{571}{622080x^5} + \frac{163879}{41803776x^6} + \mathcal{R}(x), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \mathcal{R}(x) &= -\frac{1}{2275463135232000x^{13}} \left\{ \frac{25799409925}{8} + \frac{215497370169}{2} \left(x - \frac{1}{2} \right) \right. \\ &\quad + 627676689198 \left(x - \frac{1}{2} \right)^2 + 1958303670708 \left(x - \frac{1}{2} \right)^3 \\ &\quad + 3296254174710 \left(x - \frac{1}{2} \right)^4 + 2807022408120 \left(x - \frac{1}{2} \right)^5 \\ &\quad \left. + 951982839360 \left(x - \frac{1}{2} \right)^6 \right\} < 0 \quad \text{for } x \geq \frac{1}{2}. \end{aligned}$$

Hence, upon identifying the coefficients of x^{-k-1} ($1 \leq k \leq 5$) in (3.7) as $(-k)\gamma_k$, we see that (3.4) holds for $x \geq 1/2$. This completes the proof. \square

Remark 3.1. *Noting that*

$$\begin{aligned} \sum_{k=4}^7 \gamma_k x^{-k} &= -\frac{571}{2488320x^4} - \frac{163879}{209018880x^5} + \frac{5246819}{75246796800x^6} + \frac{534703531}{902961561600x^7} \\ &= -\frac{117938613 + 800398812(x - \frac{1}{2}) + 1018764000(x - \frac{1}{2})^2 + 207204480(x - \frac{1}{2})^3}{902961561600x^7} < 0 \end{aligned}$$

holds for $x \geq 1/2$, we obtain from the right-hand side of (3.3) that

$$V(x) < 1 - \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3}, \quad x \geq \frac{1}{2}. \quad (3.8)$$

Remark 3.2. *Replacement of x by $n/2$ in (3.3) yields the following double inequality for Ω_n :*

$$\begin{aligned} \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(1 - \frac{1}{6n} + \frac{1}{72n^2} + \frac{139}{6480n^3} - \frac{571}{155520n^4} - \frac{163879}{6531840n^5} \right) &< \Omega_n \\ &< \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(1 - \frac{1}{6n} + \frac{1}{72n^2} + \frac{139}{6480n^3} - \frac{571}{155520n^4} - \frac{163879}{6531840n^5} \right. \\ &\quad \left. + \frac{5246819}{1175731200n^6} + \frac{534703531}{7054387200n^7} \right). \end{aligned} \quad (3.9)$$

From the right-hand side of (1.11) it follows that

$$\Omega_n \approx \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2}, \quad n \rightarrow \infty. \quad (3.10)$$

Theorem 3.2 develops the approximation formula (3.10) to produce a complete asymptotic expansion.

Theorem 3.2. *The following asymptotic expansion holds:*

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2} \exp \left(\sum_{j=2}^{\infty} \frac{a_j}{n^j} \right), \quad n \rightarrow \infty, \quad (3.11)$$

where the coefficients a_j ($j \geq 2$) are given by

$$a_j = \frac{(-1)^{j-1}}{2j \cdot 3^j} - \frac{2^j B_{j+1}}{j(j+1)} \quad (3.12)$$

and B_j denote the Bernoulli numbers.

Proof. Noting that $\Omega_{2x} = \pi^x / \Gamma(x+1)$, we find that

$$x \ln x - x + \ln(\sqrt{2\pi x}) - \ln \Gamma(x+1) + \frac{1}{2} \ln \left(1 + \frac{1}{6x} \right) \sim \sum_{j=2}^{\infty} \frac{a_j}{(2x)^j} \quad (3.13)$$

as $x \rightarrow \infty$, where a_j ($j \geq 2$) are real numbers to be determined.

Stirling's series for the gamma function is given (see [25, p. 140]) by

$$\ln \Gamma(x+1) \sim x \ln x - x + \ln(\sqrt{2\pi x}) + \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)x^j}, \quad x \rightarrow \infty. \quad (3.14)$$

The Maclaurin expansion of $\ln(1+t)$ with $t = 1/(6x)$ gives

$$\ln\left(1 + \frac{1}{6x}\right) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j \cdot 6^j x^j}. \quad (3.15)$$

Substitution of (3.14) and (3.15) into (3.13) yields

$$\sum_{j=2}^{\infty} \left(\frac{(-1)^{j-1}}{2j \cdot 6^j} - \frac{B_{j+1}}{j(j+1)} \right) \frac{1}{x^j} \sim \sum_{j=2}^{\infty} \frac{a_j}{2^j} \frac{1}{x^j}. \quad (3.16)$$

Equating coefficients of equal powers of x in (3.16) we obtain (3.12). This completes the proof. \square

From (3.11), we find the following explicit asymptotic expansion:

$$\begin{aligned} \Omega_n \sim \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2} \exp \left(-\frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{648n^4} - \frac{85}{3402n^5} - \frac{1}{8748n^6} \right. \\ \left. + \frac{1667}{21870n^7} - \frac{1}{104976n^8} - \frac{1679605}{3897234n^9} - \frac{1}{1180980n^{10}} + \dots \right). \end{aligned} \quad (3.17)$$

By Lemma 2.1, we then obtain the following asymptotic expansion:

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(\sum_{j=0}^{\infty} \frac{b_j}{n^j} \right), \quad n \rightarrow \infty, \quad (3.18)$$

where the coefficients b_j are given by

$$b_0 = 1, \quad b_j = \frac{1}{j} \sum_{k=1}^j k a_k b_{j-k}, \quad j \geq 1, \quad (3.19)$$

and a_j are given in (3.12). This produces the expansion

$$\begin{aligned} \Omega_n \sim \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2} \left\{ 1 - \frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{864n^4} - \frac{5261}{204120n^5} \right. \\ \left. + \frac{6889}{20995200n^6} + \frac{125549}{1632960n^7} - \dots \right\}. \end{aligned} \quad (3.20)$$

The expansion in (3.20) motivated us to establish the following double inequality for Ω_n .

Theorem 3.3. For $n \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(1 - \frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{864n^4} - \frac{5261}{204120n^5} \right) < \Omega_n \\ < \frac{1}{\sqrt{\pi(n + \frac{1}{3})}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(1 - \frac{1}{36n^2} + \frac{23}{810n^3} \right). \end{aligned} \quad (3.21)$$

Proof. It suffices to show that

$$F(x) > 0 \quad \text{and} \quad f(x) < 0 \quad \text{for} \quad x \geq \frac{1}{2},$$

where

$$F(x) = x \ln x - x + \ln(\sqrt{2\pi x}) - \ln \Gamma(x+1) + \frac{1}{2} \ln \left(1 + \frac{1}{6x}\right) - \ln \left(1 - \frac{1}{36(2x)^2} + \frac{23}{810(2x)^3} - \frac{1}{864(2x)^4} - \frac{5261}{204120(2x)^5}\right)$$

and

$$f(x) = x \ln x - x + \ln(\sqrt{2\pi x}) - \ln \Gamma(x+1) + \frac{1}{2} \ln \left(1 + \frac{1}{6x}\right) - \ln \left(1 - \frac{1}{36(2x)^2} + \frac{23}{810(2x)^3}\right).$$

Differentiating $f(x)$ and applying the left-hand side of (2.7), we obtain that for $x \geq 1/2$,

$$\begin{aligned} f'(x) &= \ln x + \frac{1}{2x} - \psi(x+1) - \frac{6480x^3 - 693x - 115 + 1080x^2}{2x(6x+1)(6480x^3 - 45x + 23)} \\ &> \frac{1}{12x^2} - \frac{1}{120x^4} - \frac{6480x^3 - 693x - 115 + 1080x^2}{2x(6x+1)(6480x^3 - 45x + 23)} \\ &= \frac{1350x^3 + 500x^2 - 93x - 23}{120x^4(6x+1)(6480x^3 - 45x + 23)} > 0. \end{aligned}$$

Hence, $f(x)$ is strictly increasing for $x \geq 1/2$, and we have

$$f(x) < \lim_{t \rightarrow \infty} f(t) = 0 \quad \text{for } x \geq \frac{1}{2}.$$

Differentiating $F(x)$ and applying the right-hand side of (2.7), we obtain that for $x \geq 1/2$,

$$\begin{aligned} F'(x) &= \ln x + \frac{1}{2x} - \psi(x+1) \\ &\quad - \frac{3(4354560x^5 - 465696x^3 - 62160x^2 + 212645x + 31566 + 725760x^4)}{2x(6x+1)(13063680x^5 - 90720x^3 + 46368x^2 - 945x - 10522)} \\ &< \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} \\ &\quad - \frac{3(4354560x^5 - 465696x^3 - 62160x^2 + 212645x + 31566 + 725760x^4)}{2x(6x+1)(13063680x^5 - 90720x^3 + 46368x^2 - 945x - 10522)} \\ &= -\frac{P_5(x - \frac{1}{2})}{2520x^6(6x+1)Q_5(x - \frac{1}{2})}, \end{aligned}$$

where

$$P_5(x) = \frac{9402445}{16} + \frac{29999799}{8}x + 14897652x^2 + 28984704x^3 + 23697723x^4 + 6076098x^5,$$

$$Q_5(x) = \frac{794995}{2} + 4059783x + 16239888x^2 + 32568480x^3 + 32659200x^4 + 13063680x^5.$$

Hence, $F'(x) < 0$ for $x \geq 1/2$. So, $F(x)$ is strictly decreasing for $x \geq 1/2$, and we have

$$F(x) > \lim_{t \rightarrow \infty} f(t) = 0 \quad \text{for } x \geq \frac{1}{2}.$$

This completes the proof. \square

Mortici [24, Theorem 3] improved (1.11) and obtained the following double inequality, for every integer $n \geq 3$ on the left-hand side and $n \geq 1$ on the right-hand side:

$$\frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\left(n + \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2}\right)}} < \Omega_n < \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\left(n + \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2} - \frac{139}{9720n^3}\right)}}. \quad (3.22)$$

This last inequality implies

$$\Omega_n = \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\left(n + \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2} + O\left(\frac{1}{n^3}\right)\right)}}. \quad (3.23)$$

If we now define v_n by the equality

$$\Omega_n = \frac{1}{\sqrt{\pi(n + v_n)}} \left(\frac{2\pi e}{n}\right)^{n/2}, \quad (3.24)$$

we find $v_n = \theta(n/2)$, where

$$\theta(x) = 2x \left\{ \left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \right)^2 - 1 \right\}. \quad (3.25)$$

Theorem 3.4 presents the asymptotic expansion and inequality for $\theta(x)$.

Theorem 3.4. (i) *The function $\theta(x)$ has the following asymptotic expansion:*

$$\theta(x) \sim \sum_{j=0}^{\infty} \frac{c_j}{x^j}, \quad x \rightarrow \infty, \quad (3.26)$$

with the coefficients c_j given by

$$c_j = 2\lambda_{j+1}, \quad j \in \mathbb{N}_0, \quad (3.27)$$

where

$$\lambda_0 = 1, \quad \lambda_j = \frac{1}{j} \sum_{k=1}^j \frac{2B_{k+1}}{k+1} \lambda_{j-k}, \quad j \in \mathbb{N}. \quad (3.28)$$

Namely,

$$\begin{aligned} \theta(x) \sim & \frac{1}{3} + \frac{1}{36x} - \frac{31}{3240x^2} - \frac{139}{77760x^3} + \frac{9871}{3265920x^4} + \frac{324179}{587865600x^5} - \frac{8225671}{3527193600x^6} \\ & - \frac{69685339}{169305292800x^7} + \frac{1674981058019}{502836719616000x^8} + \frac{24279707153761}{42238284447744000x^9} - \dots \end{aligned} \quad (3.29)$$

(ii) *For $x > 0$ and $m \in \mathbb{N}_0$,*

$$2x \left\{ \exp \left(\sum_{j=1}^{2m} \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right) - 1 \right\} < \theta(x) < 2x \left\{ \exp \left(\sum_{j=1}^{2m+1} \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right) - 1 \right\}. \quad (3.30)$$

Proof. It follows from [9, (3.6)] that

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \right)^2 \sim \sum_{j=0}^{\infty} \lambda_j x^{-j}, \quad x \rightarrow \infty, \quad (3.31)$$

with the coefficients λ_j given by

$$\lambda_0 = 1, \quad \lambda_j = \frac{1}{j} \sum_{k=1}^j \frac{2B_{k+1}}{k+1} \lambda_{j-k}, \quad j \geq 1.$$

Combination of (3.25) and (3.31) then gives (3.26).

Write (2.4) as

$$\exp \left(\sum_{j=1}^{2m} \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right) < \left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \right)^2 < \exp \left(\sum_{j=1}^{2m+1} \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right). \quad (3.32)$$

Combination of (3.25) and (3.32) then gives (3.30). This completes the proof. \square

Corollary 3.1. (i) *The sequence $v_n = \theta(n/2)$ has the following asymptotic expansion:*

$$v_n \sim \sum_{j=0}^{\infty} \frac{d_j}{n^j}, \quad n \rightarrow \infty, \quad (3.33)$$

with the coefficients d_j given by

$$d_j = 2^{j+1} \lambda_{j+1}, \quad j \in \mathbb{N}_0,$$

where λ_j are given in (3.28). Namely,

$$\begin{aligned} v_n \sim & \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2} - \frac{139}{9720n^3} + \frac{9871}{204120n^4} + \frac{324179}{18370800n^5} - \frac{8225671}{55112400n^6} \\ & - \frac{69685339}{1322697600n^7} + \frac{1674981058019}{1964205936000n^8} + \frac{24279707153761}{82496649312000n^9} - \dots \end{aligned} \quad (3.34)$$

(ii) *For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, we have*

$$n \exp \left(\sum_{j=1}^{2m} \frac{2^{2j-1} B_{2j}}{j(2j-1)n^{2j-1}} \right) < n + v_n < n \exp \left(\sum_{j=1}^{2m+1} \frac{2^{2j-1} B_{2j}}{j(2j-1)n^{2j-1}} \right). \quad (3.35)$$

It follows from (3.24) and (3.33) that

$$\Omega_n \sim \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi \left(n + \sum_{j=0}^{\infty} \frac{d_j}{n^j} \right)}} \quad (3.36)$$

which develops (3.23) to produce a complete asymptotic expansion.

From (3.24) and (3.35) we see that for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$\frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp \left(\sum_{j=1}^{2m+1} \frac{2^{2j-1} B_{2j}}{j(2j-1)n^{2j-1}} \right)}} < \Omega_n < \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp \left(\sum_{j=1}^{2m} \frac{2^{2j-1} B_{2j}}{j(2j-1)n^{2j-1}} \right)}}. \quad (3.37)$$

In particular, the choice $m = 1$ on the left-hand side and $m = 2$ on the right-hand side yields

$$\frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp \left(\frac{1}{3n} - \frac{2}{45n^3} + \frac{16}{315n^5} \right)}} < \Omega_n < \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp \left(\frac{1}{3n} - \frac{2}{45n^3} + \frac{16}{315n^5} - \frac{16}{105n^7} \right)}} \quad (3.38)$$

for $n \in \mathbb{N}$.

The inequalities (3.38) are sharper than the inequalities (3.22) for $n \geq 4$.

Based on the formula (3.1) we now establish a sharp inequality for Ω_n .

Theorem 3.5. For $n \in \mathbb{N}$, we have

$$\frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(1 - \frac{1}{6n+p} \right) < \Omega_n \leq \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n} \right)^{n/2} \left(1 - \frac{1}{6n+q} \right) \quad (3.39)$$

with the best possible constants

$$p = \frac{1}{2} \quad \text{and} \quad q = \frac{6\sqrt{2} - 5\sqrt{e}}{\sqrt{e} - \sqrt{2}} = 1.03056322\dots \quad (3.40)$$

Proof. If we write (3.39) as

$$p < x_n \leq q, \quad x_n = \left(1 - \frac{\Omega_n}{\frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n} \right)^{n/2}} \right)^{-1} - 6n,$$

we find that

$$x_1 = \frac{6\sqrt{2} - 5\sqrt{e}}{\sqrt{e} - \sqrt{2}} \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n = \frac{1}{2}. \quad (3.41)$$

The limit in (3.41) is obtained by using the asymptotic expansion (3.1).

In order to prove Theorem 3.5, it suffices to show that the sequence $\{x_n\}$ is strictly decreasing for $n \geq 1$. The monotonicity property of $\{x_n\}$ is obtained by considering the function $U(x)$ defined by

$$U(x) = \frac{1}{1 - V(x)} - 12x,$$

where $V(x)$ is given in (3.2). Differentiating $U(x)$ and applying (3.4) and (3.8), we obtain that for $x \geq 2$,

$$\begin{aligned} -(1 - V(x))^2 U'(x) &= 12(1 - V(x))^2 - V'(x) \\ &> 12 \left(\frac{1}{12x} - \frac{1}{288x^2} - \frac{139}{51840x^3} \right)^2 \\ &\quad - \left(\frac{1}{12x^2} - \frac{1}{144x^3} - \frac{139}{17280x^4} + \frac{571}{622080x^5} + \frac{163879}{41803776x^6} \right) \\ &= \frac{19066289 + 33264000(x-2) + 8860320(x-2)^2}{3135283200x^6} > 0. \end{aligned}$$

Hence, $U(x)$ is strictly decreasing for $x \geq 2$. We then obtain that the sequence $\{x_n\} = \{U(n/2)\}$ is strictly decreasing for $n \geq 4$.

Direct computation yields

$$x_1 = 1.030563\dots, \quad x_2 = 0.843071\dots, \quad x_3 = 0.748041\dots, \quad x_4 = 0.692684\dots$$

Consequently, the sequence $\{x_n\}$ is strictly decreasing for all $n \geq 1$. This completes the proof. \square

4. ASYMPTOTICS AND INEQUALITIES FOR $\frac{\Omega_{n-1}}{\Omega_n}$ AND $\frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}}$

It is easy to see that

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}, \quad \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} = \frac{\sqrt{\pi}(n+1)\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{(n+1+2\pi)\Gamma\left(\frac{n}{2} + 1\right)}. \quad (4.1)$$

The asymptotic expansion of these two ratios follows immediately from the well-known expansion for the ratio of two gamma functions [25, p. 141]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{k=0}^{\infty} \binom{a-b}{k} B_k^{(a-b+1)}(a) z^{-a} \quad (z \rightarrow \infty, \quad |\arg z| < \pi),$$

where $B_k^{(\nu)}(x)$ denote the generalized Bernoulli polynomials defined by

$$\left(\frac{t}{e^t-1}\right)^\nu e^{xt} = \sum_{k=0}^{\infty} B_k^{(\nu)}(x) \frac{t^k}{k!}, \quad |t| < 2\pi. \quad (4.2)$$

Hence we obtain the expansions

$$\begin{aligned} \frac{\Omega_{n-1}}{\Omega_n} &\sim \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \binom{1/2}{k} B_k^{(3/2)}(1) \left(\frac{2}{n}\right)^k \\ &= \sqrt{\frac{n}{2\pi}} \left\{ 1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6} \right. \\ &\quad \left. - \frac{39325}{262144n^7} - \frac{334477}{8388608n^8} + \dots \right\} \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} &\sim \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \sum_{k=0}^{\infty} \binom{-1/2}{k} B_k^{(1/2)}\left(\frac{1}{2}\right) \left(\frac{2}{n}\right)^k \\ &= \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \left\{ 1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3} - \frac{21}{2048n^4} - \frac{399}{8192n^5} + \dots \right\}. \end{aligned} \quad (4.4)$$

Formula (4.3) was presented by Mortici in [24, Theorem 8].

Upper and lower bounds for the ratio Ω_{n-1}/Ω_n can be obtained by replacement of x by $n/2$ in (2.12) to find

$$\sqrt{\frac{n}{2\pi}} \exp\left(\sum_{j=1}^{2m} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n}{2\pi}} \exp\left(\sum_{j=1}^{2m+1} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right). \quad (4.5)$$

Mortici [24, Theorem 7] established the following inequality:

$$\begin{aligned} \sqrt{\frac{n}{2\pi}} \exp\left(\frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{17}{112n^7}\right) &< \frac{\Omega_{n-1}}{\Omega_n} \\ &< \sqrt{\frac{n}{2\pi}} \exp\left(\frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{17}{112n^7} + \frac{31}{36n^9}\right), \quad n \in \mathbb{N}. \end{aligned} \quad (4.6)$$

We observe that the choice $m = 2$ in (4.5) yields (4.6).

Based on (4.3), Mortici [24, Theorem 9] established the following inequality:

$$\begin{aligned} \sqrt{\frac{n}{2\pi}} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5}\right) &< \frac{\Omega_{n-1}}{\Omega_n} \\ &< \sqrt{\frac{n}{2\pi}} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6}\right), \quad n \in \mathbb{N}. \end{aligned} \quad (4.7)$$

Remark 4.1. *The left-hand side of (4.6) is sharper than the left-hand side of (4.7) for $n \geq 10$. The right-hand side of (4.6) is sharper than the right-hand side of (4.7) for $n \geq 3$.*

Replacement of x by $n/2$ in (2.12) yields

$$\sqrt{\frac{2}{n}} \exp\left(-\sum_{j=1}^{2m} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) > \frac{\Gamma(\frac{n}{2} + \frac{1}{2})}{\Gamma(\frac{n}{2} + 1)} > \sqrt{\frac{2}{n}} \exp\left(-\sum_{j=1}^{2m+1} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) \quad (4.8)$$

for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. We then find from (4.1) and (4.8) that, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$\begin{aligned} \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \exp\left(-\sum_{j=1}^{2m+1} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) &< \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} \\ &< \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \exp\left(-\sum_{j=1}^{2m} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right). \end{aligned} \quad (4.9)$$

In particular, the choice $m = 0$ on the left-hand side and $m = 1$ on the right-hand side yields

$$\frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \exp\left(-\frac{1}{4n}\right) < \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \exp\left(-\frac{1}{4n} + \frac{1}{24n^3}\right) \quad (4.10)$$

for $n \in \mathbb{N}$.

Mortici [24, Theorem 12] improved the bounds in (1.9) and (1.10) as follows:

$$\sqrt{\frac{2\pi}{n+4\pi+\frac{1}{2}} + \varepsilon_1(n)} < \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \sqrt{\frac{2\pi}{n+4\pi+\frac{1}{2}} + \varepsilon_2(n)}, \quad (4.11)$$

where

$$\varepsilon_1(n) = -\frac{\frac{1}{4}\pi - 4\pi^2 + 8\pi^3}{n^3} \quad \text{and} \quad \varepsilon_2(n) = \varepsilon_1(n) + \frac{\frac{3}{8}\pi - 7\pi^2 - 12\pi^3 + 64\pi^4}{n^4}.$$

We remark that the left-hand side of (4.11) holds for $n \geq 10$, because

$$\frac{2\pi}{n+4\pi+\frac{1}{2}} + \varepsilon_1(n) \begin{cases} < 0, & 1 \leq n \leq 9, \\ > 0, & n \geq 10. \end{cases}$$

The right-hand side of (4.11) is valid for all $n \in \mathbb{N}$.

Remark 4.2. *The inequalities (4.10) are sharper than the inequalities (4.11) and, moreover, (4.10) is valid for all $n \in \mathbb{N}$ and has a simple form.*

Noting that

$$\exp\left(-\frac{1}{4n}\right) > 1 - \frac{1}{4n} \quad \text{and} \quad \exp\left(-\frac{1}{4n} + \frac{1}{24n^3}\right) < \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3}\right)$$

holds for $n \in \mathbb{N}$, we obtain the following alternative form of the double inequality in (4.10):

$$\begin{aligned} \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \left(1 - \frac{1}{4n}\right) &< \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} \\ &< \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3}\right) \end{aligned} \quad (4.12)$$

for $n \in \mathbb{N}$.

The following theorem presents a sharp inequality for $\Omega_n/(\Omega_{n-1} + \Omega_{n+1})$.

Theorem 4.1. *For $n \in \mathbb{N}$, we have*

$$\frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n+\frac{\pi}{2}-1}} \leq \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n+\frac{1}{2}}}. \quad (4.13)$$

The constants $\frac{\pi}{2}-1 = 0.5707963\dots$ and $\frac{1}{2}$ are the best possible.

Proof. First, we establish the left-hand inequality of (4.13). Elementary calculations show that this is valid for $n=1$ and $n=2$. We now prove that the left-hand inequality of (4.13) holds for $n \geq 3$. It suffices to show by appeal to (4.12) that

$$\frac{1}{\sqrt{n}} \left(1 - \frac{1}{4n}\right) > \frac{1}{\sqrt{n+c}} \quad \text{with } c = \frac{\pi}{2} - 1.$$

We find, for $n \geq 3$,

$$\left(\frac{1}{\sqrt{n}} \left(1 - \frac{1}{4n}\right)\right)^2 - \frac{1}{n+c} = \frac{121c-69+(88c-47)(n-3)+(16c-8)(n-3)^2}{16n^3(n+c)} > 0.$$

This proves the left-hand inequality of (4.13) for $n \geq 1$.

We now establish the right-hand inequality of (4.13). Elementary calculations show that this is valid for $n=1$. We now prove that the right-hand inequality of (4.13) holds for $n \geq 2$. It suffices to show by a similar appeal to (4.12) that

$$\frac{1}{\sqrt{n}} \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3}\right) < \frac{1}{\sqrt{n+\frac{1}{2}}}.$$

We find, for $n \geq 2$, that

$$\begin{aligned} & \left(\frac{1}{\sqrt{n}} \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3}\right)\right)^2 - \frac{1}{n+\frac{1}{2}} \\ &= -\frac{1}{16384n^7(2n+1)} \left(62899 + 192422(n-2) + 227104(n-2)^2 + 130656(n-2)^3 \right. \\ & \quad \left. + 36864(n-2)^4 + 4096(n-2)^5\right) < 0. \end{aligned}$$

This proves the right-hand inequality of (4.13) for $n \geq 1$.

If we write (4.13) as

$$\frac{\pi}{2} - 1 \geq y_n > \frac{1}{2}, \quad y_n = 2\pi \left(\frac{n+1+2\pi}{n+1} \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}}\right)^{-2} - n,$$

we find that

$$y_1 = \frac{\pi}{2} - 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = \frac{1}{2}. \quad (4.14)$$

The limit in (4.14) is obtained by using the asymptotic expansion (4.4). Hence, the inequality (4.13) holds for $n \geq 1$, and the constants $\frac{\pi}{2}-1$ and $\frac{1}{2}$ are the best possible. \square

Remark 4.3. *The inequalities (4.13) are sharper than the inequalities (1.9) and (1.10) for $n \geq 3$.*

5. ASYMPTOTICS AND INEQUALITIES FOR $\Omega_n^{1/n}/\Omega_{n+1}^{1/(n+1)}$

It is easy to see that

$$\frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} = \frac{\Gamma\left(\frac{n}{2} + \frac{3}{2}\right)^{1/(n+1)}}{\Gamma\left(\frac{n}{2} + 1\right)^{1/n}}.$$

We first establish the asymptotic expansion for $\Gamma(x + 3/2)^{1/(2x+1)}/\Gamma(x + 1)^{1/(2x)}$.

Theorem 5.1. *As $x \rightarrow \infty$, we have*

$$\frac{\Gamma\left(x + \frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x + 1)^{1/(2x)}} \sim (\sqrt{2\pi x})^{-\frac{1}{2x(2x+1)}} \exp\left\{\sum_{j=1}^{\infty} \frac{\mu_j}{x^j}\right\}, \quad (5.1)$$

with the coefficients μ_j given by $\mu_1 = 1/4$ and

$$\mu_j = (-1)^j \left\{ \frac{(-1)^{j+1} B_j}{2j(j-1)} - \frac{1}{2^{j+1}} + \sum_{k=0}^{j-2} \frac{k+2 - (2^{k+1} - 1)B_{k+2}}{(k+1)(k+2)2^j} \right\} \quad (j \geq 2). \quad (5.2)$$

Proof. The logarithm of gamma function has asymptotic expansion (see [20, p. 32]):

$$\ln \Gamma(x+t) \sim \left(x+t - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(t)}{n(n+1)} \frac{1}{x^n} \quad (5.3)$$

as $x \rightarrow \infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by (4.2) with $\nu = 1$. Using (5.3), we find as $x \rightarrow \infty$

$$\begin{aligned} & \frac{1}{2x+1} \ln \Gamma\left(x + \frac{3}{2}\right) - \frac{1}{2x} \ln \Gamma(x+1) + \frac{1}{2x(2x+1)} \ln \sqrt{2\pi x} \\ & \sim \frac{1}{2x+1} \left\{ (x+1) \ln x - x + \ln(\sqrt{2\pi}) + \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1}(3/2)}{j(j+1)} \frac{1}{x^j} \right\} \\ & \quad - \frac{1}{2x} \left\{ \left(x + \frac{1}{2}\right) \ln x - x + \ln(\sqrt{2\pi}) + \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1}(1)}{j(j+1)} \frac{1}{x^j} \right\} \\ & \quad + \frac{1}{2x(2x+1)} \ln \sqrt{2\pi x} \\ & \sim \frac{1}{2x+1} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1}(3/2)}{j(j+1)} \frac{1}{x^j} - \sum_{j=2}^{\infty} \frac{(-1)^j B_j(1)}{2j(j-1)} \frac{1}{x^j} + \frac{1}{2(2x+1)}. \end{aligned} \quad (5.4)$$

Noting that (when $x \geq 1/2$)

$$\frac{1}{2x+1} = \frac{1}{2x} \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2x)^j} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{2^j x^j} \quad (5.5)$$

holds, we obtain, as $x \rightarrow \infty$,

$$\begin{aligned} \frac{1}{2x+1} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1}(3/2)}{j(j+1)} \frac{1}{x^j} & \sim \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1}(3/2)}{j(j+1)} \frac{1}{x^j} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2^k x^k} \\ & \sim \sum_{j=2}^{\infty} \left\{ \sum_{k=0}^{j-2} \frac{(-1)^j B_{k+2}(3/2)}{(k+1)(k+2)2^{j-k-1}} \right\} \frac{1}{x^j}. \end{aligned} \quad (5.6)$$

Substitution of the expressions (5.5) and (5.6) into (5.4) then yields

$$\frac{\Gamma\left(x + \frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} \sim (\sqrt{2\pi x})^{-\frac{1}{2x(2x+1)}} \exp\left\{\sum_{j=1}^{\infty} \frac{\mu_j}{x^j}\right\},$$

with the coefficients μ_j given by

$$\mu_1 = \frac{1}{4}, \quad \mu_j = (-1)^j \left\{ -\frac{B_j(1)}{2j(j-1)} - \frac{1}{2^{j+1}} + \sum_{k=0}^{j-2} \frac{B_{k+2}(3/2)}{(k+1)(k+2)2^{j-k-1}} \right\} \quad \text{for } j \geq 2. \quad (5.7)$$

Noting that

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad B_n(1) = (-1)^n B_n, \quad B_n(1/2) = -(1-2^{1-n})B_n$$

holds (see [25, p. 590]), we find that (5.7) can be written as (5.2). This completes the proof. \square

From (5.1), we obtain the following explicit asymptotic expansion:

$$\frac{\Gamma\left(x + \frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} \sim (\sqrt{2\pi x})^{-\frac{1}{2x(2x+1)}} \exp\left\{\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4} - \frac{547}{11520x^5} + \frac{601}{23040x^6} - \frac{4691}{322560x^7} + \dots\right\}. \quad (5.8)$$

Replacement of x by $n/2$ in (5.8) yields

$$\frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} \sim (\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \exp\left\{\frac{1}{2n} + \frac{1}{4n^2} - \frac{11}{12n^3} + \frac{31}{24n^4} - \frac{547}{360n^5} + \frac{601}{360n^6} - \frac{4691}{2520n^7} + \dots\right\} \quad (5.9)$$

and, by Lemma 2.1, we finally obtain the following asymptotic expansion as $n \rightarrow \infty$:

$$\frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} \sim (\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \left\{1 + \frac{1}{2n} + \frac{3}{8n^2} - \frac{37}{48n^3} + \frac{115}{128n^4} - \frac{13781}{11520n^5} + \dots\right\}. \quad (5.10)$$

Formula (5.8) motivated us to establish the following double inequality for $\Gamma(x+3/2)^{1/(2x+1)}/\Gamma(x+1)^{1/(2x)}$.

Theorem 5.2. For $x \geq 1/2$,

$$\begin{aligned} (\sqrt{2\pi x})^{-\frac{1}{2x(2x+1)}} \exp\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) &< \frac{\Gamma\left(x + \frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} \\ &< (\sqrt{2\pi x})^{-\frac{1}{2x(2x+1)}} \exp\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4}\right). \end{aligned} \quad (5.11)$$

Proof. In order to prove the left-hand side of (5.11), it suffices to show that for $x \geq 1/2$,

$$\begin{aligned} \frac{1}{2x+1} \ln\left(x + \frac{1}{2}\right) + \frac{1}{2x+1} \ln \Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2x} \ln \Gamma(x+1) \\ + \frac{1}{2x(2x+1)} \ln \sqrt{2\pi x} - \left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) > 0; \end{aligned}$$

that is,

$$\begin{aligned} u(x) &:= \ln\left(x + \frac{1}{2}\right) - \ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) - \frac{1}{2x}\left(\ln\Gamma(x+1) - \ln\sqrt{2\pi x}\right) \\ &\quad - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) > 0. \end{aligned}$$

Similarly, to prove the right-hand side of (5.11), it suffices to show that for $x \geq 1/2$,

$$\begin{aligned} v(x) &:= \ln\left(x + \frac{1}{2}\right) - \ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) - \frac{1}{2x}\left(\ln\Gamma(x+1) - \ln\sqrt{2\pi x}\right) \\ &\quad - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4}\right) < 0. \end{aligned}$$

Using the inequalities (2.5) and (2.13), we obtain, for $x \geq 1/2$,

$$\begin{aligned} u(x) &> \ln x + \ln\left(1 + \frac{1}{2x}\right) - \left(\frac{1}{2}\ln x + \frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5}\right) \\ &\quad - \frac{1}{2x}\left(x \ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}\right) - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) \\ &= \ln\left(1 + \frac{1}{2x}\right) - \frac{20160x^5 - 5040x^4 - 4830x^3 - 56x^2 + 63x + 16}{40320x^6} \\ &> \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} - \frac{1}{64x^4} - \frac{20160x^5 - 5040x^4 - 4830x^3 - 56x^2 + 63x + 16}{40320x^6} \\ &= \frac{2491}{4} + \frac{8491}{2}\left(x - \frac{1}{2}\right) + 9191\left(x - \frac{1}{2}\right)^2 + 6510\left(x - \frac{1}{2}\right)^3 \\ &\quad \frac{> 0}{40320x^6} \end{aligned}$$

and

$$\begin{aligned} v(x) &< \ln x + \ln\left(1 + \frac{1}{2x}\right) - \left(\frac{1}{2}\ln x + \frac{1}{8x} - \frac{1}{192x^3}\right) \\ &\quad - \frac{1}{2x}\left(x \ln x - x + \frac{1}{12x} - \frac{1}{360x^3}\right) - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4}\right) \\ &= \ln\left(1 + \frac{1}{2x}\right) - \frac{2880x^3 - 720x^2 + 240x + 457}{5760x^4} \\ &< \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} - \frac{2880x^3 - 720x^2 + 240x + 457}{5760x^4} = -\frac{457}{5760x^4} < 0. \end{aligned}$$

This completes the proof. \square

Replacement of x by $n/2$ in (5.11) yields

$$\begin{aligned} (\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \exp\left(\frac{1}{2n} + \frac{1}{4n^2} - \frac{11}{12n^3}\right) &< \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} \\ &< (\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \exp\left(\frac{1}{2n} + \frac{1}{4n^2} - \frac{11}{12n^3} + \frac{31}{24n^4}\right), \quad n \in \mathbb{N}. \end{aligned} \quad (5.12)$$

The expansion (5.10) motivated us to establish the following sharp inequality for $\Omega_n^{1/n}/\Omega_{n+1}^{1/(n+1)}$.

Theorem 5.3. For $n \in \mathbb{N}$, we have

$$(\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \left(1 + \frac{1}{2n - \frac{4-3\pi^{1/4}}{2-\pi^{1/4}}}\right) \leq \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} < (\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \left(1 + \frac{1}{2n - \frac{3}{2}}\right). \quad (5.13)$$

The constants $\frac{4-3\pi^{1/4}}{2-\pi^{1/4}} = 0.008963\dots$ and $\frac{3}{2}$ are the best possible.

Proof. First, we establish the left-hand inequality of (5.13). Elementary calculations show that this is valid for $n = 1, 2$ and 3 . We now prove the left-hand inequality of (5.13) for $n \geq 4$. For this it suffices to show by appeal to (5.12) that

$$G(n) > 0 \quad \text{for } n \geq 4,$$

where

$$G(x) = \frac{1}{2x} + \frac{1}{4x^2} - \frac{11}{12x^3} - \ln \left(1 + \frac{1}{2x - a}\right) \quad \text{with } a = \frac{4 - 3\pi^{1/4}}{2 - \pi^{1/4}}.$$

Differentiation yields

$$G'(x) = -\frac{P_3(x-4)}{4x^4(2x+1-a)(2x-a)},$$

where

$$\begin{aligned} P_3(x) &= 29a^2 - 493a + 40 + (18a^2 - 422a + 234)x \\ &\quad + (2a^2 - 106a + 104)x^2 + (12 - 8a)x^3 > 0 \quad \text{for } x \geq 0. \end{aligned}$$

We then obtain $G'(x) < 0$ for $x \geq 4$. So, the sequence $\{G(n)\}$ is strictly decreasing for $n \geq 4$, and we have

$$G(n) > \lim_{m \rightarrow \infty} G(m) = 0 \quad \text{for } n \geq 4.$$

This proves the left-hand inequality of (5.13) for $n \geq 1$.

Now, we establish the right-hand inequality of (5.13). Elementary calculations show that this is valid for $n = 1$. We now prove that the right-hand inequality of (5.13) holds for $n \geq 2$. It suffices to show by appeal to (5.12) that

$$g(n) < 0 \quad \text{for } n \geq 2,$$

where

$$g(x) = \frac{1}{2x} + \frac{1}{4x^2} - \frac{11}{12x^3} + \frac{31}{24x^4} - \ln \left(1 + \frac{1}{2x - \frac{3}{2}}\right).$$

Differentiation yields

$$g'(x) = \frac{692 + 2211(x-2) + 2098(x-2)^2 + 606(x-2)^3}{12x^5(4x-1)(4x-3)} > 0 \quad (x \geq 2).$$

Hence, the sequence $\{g(n)\}$ is strictly increasing for $n \geq 2$, and we have

$$g(n) < \lim_{m \rightarrow \infty} g(m) = 0 \quad \text{for } n \geq 2.$$

This proves that the right-hand inequality of (5.13) for $n \geq 1$.

If we write (5.13) as

$$\frac{4 - 3\pi^{1/4}}{2 - \pi^{1/4}} \leq x_n < \frac{3}{2}, \quad x_n = 2n - \left(\frac{(\sqrt{n\pi})^{-\frac{1}{n(n+1)}} \Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} - 1 \right)^{-1},$$

we find that

$$x_1 = \frac{4 - 3\pi^{1/4}}{2 - \pi^{1/4}} \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n = \frac{3}{2}. \quad (5.14)$$

The limit in (5.14) is obtained by using the asymptotic expansion (5.10). Hence, the inequality (5.13) holds for $n \geq 1$, and the constants $\frac{4-3\pi^{1/4}}{2-\pi^{1/4}}$ and $\frac{3}{2}$ are the best possible. \square

Remark 5.1. Write (1.1) as

$$\left(\frac{4}{\pi}\right)^{1/(2n)} \leq \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} < e^{1/(2n)}. \quad (5.15)$$

The inequalities (5.13) are sharper than the inequalities (5.15) for $n \geq 2$.

REFERENCES

- [1] H. Alzer, On some inequalities for the gamma and psi functions, *Math. Comput.* 66 (1997) 373–389.
- [2] H. Alzer, Inequalities for the volume of the unit ball in \mathbb{R}^n , *J. Math. Anal. Appl.* 252 (2000) 353–363.
- [3] H. Alzer, Inequalities for the volume of the unit ball in \mathbb{R}^n , II, *Mediterr. J. Math.* 5 (2008) 395–413.
- [4] G.D. Anderson, S.-L. Qiu, A monotonicity property of the gamma function, *Proc. Amer. Math. Soc.* 125 (1997) 3355–3362.
- [5] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Special functions of quasiconformal theory, *Exposition. Math.* 7 (1989) 97–136.
- [6] T. Ban, C.-P. Chen, New inequalities for the volume of the unit ball in \mathbb{R}^n , *J. Math. Inequal.* 11 (2) (2017) 527–542.
- [7] J. Böhm, E. Hertel, *Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung*, Birkhäuser, Basel, 1981.
- [8] K.H. Borgwardt, *The Simplex Method*, Springer, Berlin, 1987.
- [9] C.-P. Chen, Inequalities and asymptotic expansions associated with the Ramanujan and Nemes formulas for the gamma function, *Appl. Math. Comput.* 261 (2015) 337–350.
- [10] C.-P. Chen, N. Elezović, L. Vukšić, Asymptotic formulae associated with the Wallis power function and digamma function, *J. Classical Anal.* 2 (2013) 151–166.
- [11] C.-P. Chen, L. Lin, Inequalities for the volume of the unit ball in \mathbb{R}^n , *Mediterr. J. Math.* 11 (2014) 299–314.
- [12] C.-P. Chen, R.B. Paris, Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function, *Appl. Math. Comput.* 250 (2015) 514–529.
- [13] J. Dubourdieu, Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes, *Compositio Math.* 7 (1939) 96–111 (in French).
- [14] B.-N. Guo, F. Qi, Monotonicity and logarithmic convexity relating to the volume of the unit ball, *Optim. Lett.* 7 (2013) 1139–1153.
- [15] H. van Haeringen, Completely monotonic and related functions, *J. Math. Anal. Appl.* 204 (1996) 389–408.
- [16] D.A. Klain, G.-C. Rota, A continuous analogue of Sperner’s theorem, *Comm. Pure Appl. Math.* 50 (1997) 205–223.
- [17] S. Koumandos, Remarks on some completely monotonic functions, *J. Math. Anal. Appl.* 324 (2006) 1458–1461.
- [18] S. Koumandos, H.L. Pedersen, Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function, *J. Math. Anal. Appl.* 355 (2009) 33–40.
- [19] D. Lu, P. Zhang, A new general asymptotic formula and inequalities involving the volume of the unit ball, *J. Number Theory* 170 (2017) 302–314.
- [20] Y.L. Luke, *The Special Functions and their Approximations*, vol. I, Academic Press, New York, 1969.
- [21] M. Merkle, Gurland’s ratio for the gamma function, *Comp. Math. Appl.* 49 (2005) 389–406.
- [22] C. Mortici, Monotonicity properties of the volume of the unit ball in \mathbb{R}^n , *Optim. Lett.* 4 (2010) 457–464.
- [23] C. Mortici, Estimates of the function and quotient by minc-Sathre, *Appl. Math. Comput.* 253 (2015) 52–60.
- [24] C. Mortici, Series associated to some expressions involving the volume of the unit ball and applications, *Appl. Math. Comput.* 294 (2017) 121–138.
- [25] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

- [26] R.B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Cambridge University Press, Cambridge, 2001.
- [27] N.M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Wiley, New York, 1996.
- [28] L. Yin, L.-G. Huang, Some inequalities for the volume of the unit ball, *J. Class. Anal.* 6 (1) (2015) 39–46.

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