

## On extensions of two results due to Ramanujan

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**Abstract:** The aim in this note is to provide a generalization of an interesting entry in Ramanujan's notebooks that relate sums involving the derivatives of a function  $\varphi(t)$  evaluated at 0 and 1. The generalization obtained is derived with the help of expressions for the sum of terminating  ${}_3F_2$  hypergeometric functions of argument equal to 2, recently obtained by Kim et. al. [Two results for the terminating  ${}_3F_2(2)$  with applications, Bulletin of the Korean Mathematical Society 2012; 49: 621-633]. Several special cases are given. In addition we generalize a summation formula to include integral parameter differences.

**Key words:** Hypergeometric series, Ramanujan's sum, sums of Hermite polynomials

### 1. Introduction

Two of the many interesting results stated by Ramanujan in his notebooks are the following theorems, which appear as Entry 8 [1, p. 51] and Entry 20 [1, p. 36], expressing an infinite sum of derivatives of a function  $\varphi(t)$  at the origin to another infinite sum of its derivatives evaluated at  $t = 1$ .

**Entry 8.** Let  $\varphi(t)$  be analytic for  $|t - 1| < R$ , where  $R > 1$ . Suppose that  $a$  and  $\varphi(t)$  are such that the order of summation in

$$\sum_{k=0}^{\infty} \frac{2^k (a)_k}{(2a)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!} (-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{2^k (a)_k \varphi^{(k)}(0)}{(2a)_k k!} = \sum_{k=0}^{\infty} \frac{\varphi^{(2k)}(1)}{2^{2k} (a + \frac{1}{2})_k k!}. \quad (1.1)$$

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**Entry 20.** Let  $\varphi(t) = \sum_{k=0}^{\infty} \varphi^{(k)}(1)(t-1)^k/k!$  be analytic for  $|t-1| < R$ , where  $R > 1$ . Suppose that  $a$  and  $b$  are complex parameters such that the order of summation in

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!} (-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{(a)_k \varphi^{(k)}(0)}{(b)_k k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (b-a)_k}{(b)_k k!} \varphi^{(k)}(1). \tag{1.2}$$

Berndt [1] pointed out that Entry 8 can be established with the help of the results

$${}_2F_1 \left[ \begin{matrix} -2n, a \\ 2a \end{matrix} ; 2 \right] = \frac{(\frac{1}{2})_n}{(a + \frac{1}{2})_n}, \quad {}_2F_1 \left[ \begin{matrix} -2n-1, a \\ 2a \end{matrix} ; 2 \right] = 0,$$

for nonnegative integer  $n$ , where  $(a)_n = \Gamma(a+n)/\Gamma(a)$  denotes the Pochhammer symbol.

In [4], each of the above theorems was generalized. The result in (1.1) was extended by replacing the denominatorial parameter  $2a$  by  $2a + j$ , where  $j = 0, \pm 1, \dots, \pm 5$ . The second result in (1.2) was extended by the inclusion of an additional pair of numeratorial and denominatorial parameters differing by unity to produce the following theorem.

**Theorem 1.1.** Let  $\varphi(t)$  be analytic for  $|t-1| < R$ , where  $R > 1$ . Suppose that  $a, b, d$ , and  $\varphi(t)$  are such that the order of summation in

$$\sum_{k=0}^{\infty} \frac{(a)_k (d+1)_k}{(b)_k (d)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!} (-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{(a)_k (d+1)_k}{(b)_k (d)_k} \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (b-a-1)_k (f+1)_k}{(b)_k (f)_k} \frac{\varphi^{(k)}(1)}{k!}, \tag{1.3}$$

where  $f = d(b-a-1)/(d-a)$ .

This result was extended to the case where a pair of numeratorial and denominatorial parameters differs by a positive integer  $m$  to produce

$$\sum_{k=0}^{\infty} \frac{(a)_k (d+m)_k}{(b)_k (d)_k} \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (b-a-m)_k ((\xi_m+1))_k}{(b)_k ((\xi_m))_k} \frac{\varphi^{(k)}(1)}{k!},$$

where the  $\xi_1, \dots, \xi_m$  are the zeros of a certain polynomial of degree  $m$ .

In this note we shall similarly extend the result in (1.1) (when the parameter  $2a$  is replaced by  $2a + 1$ ) by the inclusion of a pair of numeratorial and denominatorial parameters differing by unity. For this we shall

require the summations of a  ${}_3F_2$  hypergeometric function of argument equal to 2 obtained\* in [3, Theorem 2]

$${}_3F_2 \left[ \begin{matrix} -2n, a, d+1 \\ 2a+1, d \end{matrix} ; 2 \right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, \tag{1.4}$$

and

$${}_3F_2 \left[ \begin{matrix} -2n-1, a, d+1 \\ 2a+1, d \end{matrix} ; 2 \right] = \frac{(1-2a/d)}{2a+1} \frac{(\frac{3}{2})_n}{(a+\frac{3}{2})_n}, \tag{1.5}$$

for nonnegative integer  $n$ . Several applications are presented in Section 3.

In the final section we generalize the result given in [1, p. 25] as Entry 9:

**Entry 9.** *If  $Re(c-a) > 0$ , then*

$$\sum_{k=1}^{\infty} \frac{(a)_k}{k(c)_k} = \psi(c) - \psi(c-a), \tag{1.6}$$

where  $\psi(x)$  denotes the logarithmic derivative of  $\Gamma(x)$ .

We extend this summation to include additional numeratorial and denominatorial parameters differing by positive integers. To achieve this we make use of the generalized Karlsson–Minton summation formula for a  ${}_{r+2}F_{r+1}$  hypergeometric function of unit argument.

## 2. Generalization of Ramanujan’s result (1.1)

The result to be established in this section is given by the following theorem.

**Theorem 2.1.** *Let  $\varphi(t)$  be analytic for  $|t-1| < R$ , where  $R > 1$ . Suppose that  $a, d$ , and  $\varphi(t)$  are such that the order of summation in*

$$\sum_{k=0}^{\infty} \frac{2^k (a)_k (d+1)_k}{(2a+1)_k (d)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!} (-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{2^k (a)_k (d+1)_k \varphi^{(k)}(0)}{(2a+1)_k (d)_k k!} = \sum_{k=0}^{\infty} \frac{\varphi^{(2k)}(1)}{2^{2k} (a+\frac{1}{2})_k k!} - \frac{(1-2a/d)}{2a+1} \sum_{k=0}^{\infty} \frac{\varphi^{(2k+1)}(1)}{2^{2k} (a+\frac{3}{2})_k k!}. \tag{2.1}$$

*Proof.* Since  $\varphi(t)$  is analytic for  $|t-1| < R$ , we have

$$\varphi^{(k)}(0) = \sum_{n=k}^{\infty} \frac{(-1)^n (-n)_k}{n!} \varphi^{(n)}(1)$$

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\*It should be noted that the right-hand side of (1.4) is independent of the parameter  $d$ .

by suitable differentiation of the associated Taylor series. Then

$$\begin{aligned} S &:= \sum_{k=0}^{\infty} \frac{2^k(a)_k(d+1)_k}{(2a+1)_k(d)_k} \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{2^k(a)_k(d+1)_k}{(2a+1)_k(d)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n(-n)_k}{n!} \varphi^{(n)}(1) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n)}(1)}{n!} \sum_{k=0}^n \frac{2^k(a)_k(d+1)_k(-n)_k}{(2a+1)_k(d)_k k!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n)}(1)}{n!} {}_3F_2 \left[ \begin{matrix} -n, a, d+1 \\ 2a+1, d \end{matrix}; 2 \right] \end{aligned}$$

upon inversion of the order of summation by hypothesis.

If we now separate the above sum into terms involving even and odd  $n$ , we obtain

$$S = \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(1)}{(2n)!} {}_3F_2 \left[ \begin{matrix} -2n, a, d+1 \\ 2a+1, d \end{matrix}; 2 \right] - \sum_{n=0}^{\infty} \frac{\varphi^{(2n+1)}(1)}{(2n+1)!} {}_3F_2 \left[ \begin{matrix} -2n-1, a, d+1 \\ 2a+1, d \end{matrix}; 2 \right].$$

Finally, using the summations in (1.4) and (1.5) and noting that  $(2n)! = 2^{2n}(\frac{1}{2})_n n!$ ,  $(2n+1)! = 2^{2n}(\frac{3}{2})_n n!$ , we easily arrive at the right-hand side of (2.1). This completes the proof of the theorem.  $\square$

When  $d = 2a$  it is seen that (2.1) reduces to Ramanujan’s result in (1.1).

### 3. Examples of Theorem 2.1

In this section, we provide some examples of different choices for the function  $\varphi(t)$  appearing in (2.1). Throughout this section we let  $k$  denote a nonnegative integer.

(a) First we consider the simplest choice with  $\varphi(t) = \exp(xt)$ , where  $x$  is an arbitrary variable (independent of  $t$ ). Then  $\varphi^{(k)}(t) = x^k \varphi(t)$ , which satisfies the conditions for the validity of (2.1). Substitution of the derivatives into (2.1) and identification of the resulting series as hypergeometric functions immediately yields

$$e^{-x} {}_2F_2 \left[ \begin{matrix} a, d+1 \\ 2a+1, d \end{matrix}; 2x \right] = {}_0F_1 \left[ \begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{1}{4}x^2 \right] - \frac{(1-2a/d)x}{2a+1} {}_0F_1 \left[ \begin{matrix} - \\ a + \frac{3}{2} \end{matrix}; \frac{1}{4}x^2 \right], \tag{3.1}$$

which is a result established by a different method in [9, Theorem 2]. In addition, it is interesting to observe that, since

$${}_0F_1 \left[ \begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{1}{4}x^2 \right] = \Gamma(a + \frac{1}{2}) (\frac{1}{2}x)^{\frac{1}{2}-a} I_{a-\frac{1}{2}}(x),$$

where  $I_\nu$  is the modified Bessel function of the first kind, the result (3.1) can also be written in terms of  $I_\nu$ .

(b) If we let  $\varphi(t) = \cosh(xt)$ , we have  $\varphi^{(2k)}(t) = x^{2k} \cosh(xt)$  and  $\varphi^{(2k+1)}(t) = x^{2k+1} \sinh(xt)$ . Then (2.1), after a little simplification making use of the identity

$$(a)_{2k} = (\frac{1}{2}a)_k (\frac{1}{2}a + \frac{1}{2})_k 2^{2k},$$

and letting  $d \rightarrow 2d$ , reduces to

$${}_3F_4 \left[ \begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, d+1 \\ \frac{1}{2}, a + \frac{1}{2}, a+1, d \end{matrix}; x^2 \right] = \cosh x {}_0F_1 \left[ \begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{1}{4}x^2 \right] - \frac{(1-a/d)}{2a+1} x \sinh x {}_0F_1 \left[ \begin{matrix} - \\ a + \frac{3}{2} \end{matrix}; \frac{1}{4}x^2 \right]. \tag{3.2}$$

(c) If  $\varphi(t) = (x - t)^{-b}$ , where  $b$  is an arbitrary parameter and  $x > 2$ , then

$$\varphi^{(k)}(t) = \frac{(b)_k}{(x - t)^{b+k}}.$$

From (2.1), we therefore find

$$\begin{aligned} \left(\frac{x}{x-1}\right)^{-b} \sum_{k=0}^{\infty} \frac{2^k (a)_k (b)_k (d+1)_k}{(2a+1)_k (d)_k k!} x^{-k} &= \sum_{k=0}^{\infty} \frac{2^{-2k} (b)_{2k}}{(a + \frac{1}{2})_k k!} (x-1)^{-2k} \\ &\quad - \frac{(1-2a/d)}{2a+1} \sum_{k=0}^{\infty} \frac{2^{-2k} (b)_{2k+1}}{(a + \frac{3}{2})_k k!} (x-1)^{-2k-1}, \end{aligned}$$

which yields

$$\begin{aligned} \left(\frac{x}{x-1}\right)^{-b} {}_3F_2 \left[ a, b, d+1; \frac{2}{x} \right] &= {}_2F_1 \left[ \frac{1}{2}b, \frac{1}{2}b + \frac{1}{2}; \frac{1}{(x-1)^2} \right] \\ &\quad - \frac{(1-2a/d)b}{2a+1} (x-1)^{-1} {}_2F_1 \left[ \frac{1}{2}b + \frac{1}{2}, \frac{1}{2}b + 1; \frac{1}{(x-1)^2} \right]. \end{aligned}$$

If we put  $z = x/(1+x)$ , this last result becomes

$$\begin{aligned} (1+z)^{-b} {}_3F_2 \left[ a, b, d+1; \frac{2z}{1+z} \right] &= {}_2F_1 \left[ \frac{1}{2}b, \frac{1}{2}b + \frac{1}{2}; z^2 \right] - \frac{(1-2a/d)b}{2a+1} z {}_2F_1 \left[ \frac{1}{2}b + \frac{1}{2}, \frac{1}{2}b + 1; z^2 \right], \end{aligned} \tag{3.3}$$

which has been obtained by different methods in [3, Theorem 3].

(d) Finally, with  $\varphi(t) = \exp(-x^2 t^2/4)$ , we have [8, p. 442]

$$\varphi^{(k)}(t) = (-1)^k (x/2)^k e^{-x^2 t^2/4} H_k(xt/2),$$

where  $H_k$  is the Hermite polynomial of order  $k$ . Since  $H_{2k}(0) = (-1)^k (2k)!/k!$  and  $H_{2k+1}(0) = 0$ , it follows from (2.1) that

$$\begin{aligned} e^{x^2/4} {}_3F_3 \left[ \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, d+1; -x^2 \right] &= \sum_{k=0}^{\infty} \frac{(x/4)^{2k}}{(a + \frac{1}{2})_k k!} H_{2k}(x/2) + \frac{(1-a/d)}{a + \frac{1}{2}} \sum_{k=0}^{\infty} \frac{(x/4)^{2k+1}}{(a + \frac{3}{2})_k k!} H_{2k+1}(x/2) \end{aligned} \tag{3.4}$$

provided  $a \neq -\frac{1}{2}, -\frac{3}{2}, \dots$ , where we have put  $d \rightarrow 2d$ .

The above series involving the Hermite polynomials can be expressed in terms of  ${}_2F_2$  functions since [4, Eqs. (36), (37)]

$$\sum_{k=0}^{\infty} \frac{(x/4)^{2k}}{(a + \frac{1}{2})_k k!} H_{2k}(x/2) = e^{x^2/4} {}_2F_2 \left[ \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}; -x^2 \right],$$

$$\sum_{k=0}^{\infty} \frac{(x/4)^{2k}}{(a + \frac{3}{2})_k k!} H_{2k+1}(x/2) = x e^{x^2/4} {}_2F_2 \left[ \begin{matrix} \frac{1}{2}a + 1, \frac{1}{2}a + \frac{3}{2} \\ a + \frac{3}{2}, a + 2 \end{matrix}; -x^2 \right],$$

to yield

$${}_3F_3 \left[ \begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, d + 1 \\ a + \frac{1}{2}, a + 1, d \end{matrix}; -x^2 \right] = {}_2F_2 \left[ \begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2} \\ a, a + \frac{1}{2} \end{matrix}; -x^2 \right] + \frac{x^2(1 - a/d)}{4a + 2} {}_2F_2 \left[ \begin{matrix} \frac{1}{2}a + 1, \frac{1}{2}a + \frac{3}{2} \\ a + \frac{3}{2}, a + 2 \end{matrix}; -x^2 \right].$$

We remark that this last result can be derived alternatively by writing  $(d + 1)_k / (d)_k = 1 + k/d$  in the series expansion of the  ${}_3F_3$  function combined with use of the result for contiguous  ${}_2F_2$  functions [4]:

$${}_2F_2 \left[ \begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix}; z \right] - {}_2F_2 \left[ \begin{matrix} \alpha, \beta \\ \gamma, \delta + 1 \end{matrix}; z \right] = \frac{\alpha\beta z}{\gamma\delta(\delta + 1)} {}_2F_2 \left[ \begin{matrix} \alpha + 1, \beta + 1 \\ \gamma + 1, \delta + 2 \end{matrix}; z \right].$$

Finally, the representation (3.4) may be contrasted with the more general result obtained from Theorem 1 with  $\varphi(t) = \exp(-x^2 t^2/4)$  given in [4, Eq. (40)] (with  $x \rightarrow 2x$  and  $d \rightarrow 2d$ ):

$$e^{x^2} {}_3F_3 \left[ \begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, d + 1 \\ \frac{1}{2}b, \frac{1}{2}b + \frac{1}{2}, d \end{matrix}; -x^2 \right] = \sum_{k=0}^{\infty} \frac{(b - a - 1)_k (f + 1)_k}{(b)_k (f)_k} x^k H_k(x),$$

where  $f = 2d(b - a - 1) / (2d - a)$ .

#### 4. Extension of the summation (1.6)

We employ the usual convention of writing the finite sequence of parameters  $(a_1, \dots, a_p)$  simply by  $(a_p)$  and the product of  $p$  Pochhammer symbols by  $((a_p))_k \equiv (a_1)_k \dots (a_p)_k$ . In order to derive our extension of Ramanujan’s sum (1.6) we make use of the generalized Karlsson–Minton summation theorem given below.

**Theorem 4.1** *Let  $(m_r)$  be a sequence of positive integers and  $m := m_1 + \dots + m_r$ . The generalized Karlsson–Minton summation theorem is given by [6, 7]*

$${}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, (d_r + m_r) \\ c, (d_r) \end{matrix}; 1 \right] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \sum_{k=0}^m \frac{(-1)^k (a)_k (b)_k C_{k,r}}{(1 + a + b - c)_k} \tag{4.1}$$

provided  $\Re(c - a - b) > m$ .

The coefficients  $C_{k,r}$  appearing in (4.1) are defined for  $0 \leq k \leq m$  by

$$C_{k,r} = \frac{1}{\Lambda} \sum_{j=k}^m \sigma_j \mathbf{S}_j^{(k)}, \quad \Lambda = (d_1)_{m_1} \dots (d_r)_{m_r}, \tag{4.2}$$

with  $C_{0,r} = 1$ ,  $C_{m,r} = 1/\Lambda$ .  $\mathbf{S}_j^{(k)}$  denotes the Stirling numbers of the second kind and the  $\sigma_j$  ( $0 \leq j \leq m$ ) are generated by the relation

$$(d_1 + x)_{m_1} \dots (d_r + x)_{m_r} = \sum_{j=0}^m \sigma_j x^j. \tag{4.3}$$

In [5], an alternative representation for the coefficients  $C_{k,r}$  is given as the terminating hypergeometric series of unit argument

$$C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[ -k, \begin{matrix} (d_r + m_r) \\ (d_r) \end{matrix}; 1 \right]. \tag{4.4}$$

When  $r = 1$ , with  $d_1 = d$ ,  $m_1 = m$ , Vandermonde’s summation theorem [10, p. 243] can be used to show that

$$C_{k,1} = \binom{m}{k} \frac{1}{(d)_k}. \tag{4.5}$$

Our extension of Ramanujan’s summation in (1.6) is given by the following theorem.

**Theorem 4.2** *Let  $(m_r)$  be a sequence of positive integers and  $m := m_1 + \dots + m_r$ . Then, provided  $\text{Re}(c - a) > m$ , we have*

$$\sum_{k=1}^{\infty} \frac{(a)_k ((d_r + m_r))_k}{(c)_k ((d_r))_k k} = \psi(c) - \psi(c - a) + \sum_{k=1}^m \frac{(-1)^k (a)_k \Gamma(k) C_{k,r}}{(1 + a - c)_k}, \tag{4.6}$$

where  $C_{k,r}$  are the coefficients defined in (4.3) and (4.4).

*Proof.* We follow the method of proof given in [1, p. 25]. If we differentiate logarithmically the left-hand side of (4.1) with respect to  $b$  and then set  $b = 0$ , making use of the simple fact that

$$\left. \frac{d}{dx} (x)_k \right|_{x=0} = (k - 1)!, \quad k \geq 1,$$

we immediately obtain

$$\sum_{k=1}^{\infty} \frac{(a)_k ((d_r + m_r))_k}{(c)_k ((d_r))_k k}$$

when  $\text{Re}(c - a) > m$ . Proceeding in a similar manner for the right-hand side of (4.1), we obtain

$$\psi(c) - \psi(c - a) + \sum_{k=1}^m \frac{(-1)^k (a)_k \Gamma(k) C_{k,r}}{(1 + a - c)_k}.$$

This completes the proof of the theorem.  $\square$

When  $r = 1$ , the coefficients  $C_{k,1}$  are given by (4.5) and we obtain the summation

$$\sum_{k=1}^{\infty} \frac{(a)_k (d + m)_k}{(c)_k (d)_k k} = \psi(c) - \psi(c - a) + m! \sum_{k=1}^m \frac{(-1)^k (a)_k}{k(m - k)! (1 + a - c)_k (d)_k} \tag{4.7}$$

provided  $\text{Re}(c - a) > m$ .

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