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ON THE WILKER AND HUYGENS–TYPE INEQUALITIES

CHAO-PING CHEN* AND RICHARD B. PARIS

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Abstract. Chen and Cheung [3] established sharp Wilker and Huygens-type inequalities. These authors also proposed three conjectures on Wilker and Huygens-type inequalities. In this paper, we consider these conjectures. We also present sharp Wilker and Huygens-type inequalities.

1. Introduction

Wilker [18] proposed the following two open problems:
(a) Prove that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.1}$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

for $0 < x < \pi/2$. In [17], the inequality (1.1) was proved, and the following inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x, \qquad 0 < x < \frac{\pi}{2} \tag{1.2}$$

was also established, where the constants $(2/\pi)^4$ and $\frac{8}{45}$ are the best possible.

The Wilker-type inequalities (1.1) and (1.2) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs, various generalizations and improvements (cf. [1, 2, 3, 6, 8, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23, 25, 26, 27, 28] and the references cited therein).

A related inequality that is of interest to us is Huygens' inequality [9], which asserts that

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \qquad 0 < |x| < \frac{\pi}{2}. \tag{1.3}$$

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* Corresponding author.

Wu and Srivastava [21, Lemma 3] established another inequality

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \qquad 0 < |x| < \frac{\pi}{2}. \tag{1.4}$$

Neuman and Sándor [15, Theorem 2.3] proved that for $0 < |x| < \pi/2$,

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} < \frac{1}{2}\left(\frac{x}{\sin x} + \cos x\right). \tag{1.5}$$

By multiplying both sides of inequality (1.5) by $x/\sin x$, we obtain that for $0 < |x| < \pi/2$,

$$\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right] > \frac{2(x/\sin x) + x/\tan x}{3} > 1. \tag{1.6}$$

Chen and Sándor [6] established the following inequality chain:

$$\frac{(\sin x/x)^2 + \tan x/x}{2} > \left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right) > \frac{2\,(\sin x/x) + \tan x/x}{3}$$
$$> \left(\frac{\sin x}{x}\right)^{2/3}\left(\frac{\tan x}{x}\right)^{1/3} > \frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right] > \frac{2(x/\sin x) + x/\tan x}{3} > 1 \tag{1.7}$$

for $0 < |x| < \pi/2$.

In analogy with (1.2), Chen and Cheung [3] established sharp Wilker and Huygens-type inequalities. For example, these authors proved that for $0 < x < \pi/2$,

$$2 + \frac{8}{45}x^4 + \frac{16}{315}x^5\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^4 + \left(\frac{2}{\pi}\right)^6 x^5 \tan x, \tag{1.8}$$

where the constants $\frac{16}{315}$ and $(2/\pi)^6$ are best possible,

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^3\tan x, \tag{1.9}$$

where the constant $\frac{2}{45}$ is best possible, and

$$3 + \frac{3}{20}x^3\tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3\tan x, \tag{1.10}$$

where the constants $\frac{3}{20}$ and $(2/\pi)^4$ are best possible.

In view of (1.8), (1.9) and (1.10), Chen and Cheung [3] posed the following conjectures.

CONJECTURE 1.1. For $0 < x < \pi/2$ and $n \geqslant 3$,

$$2 + \sum_{k=3}^{n} \frac{(2(2^{2k}-1)|B_{2k}| - (-1)^k)2^{2k-1}}{(2k)!}x^{2k-2}$$

$$+\frac{\left(2(2^{2n+2}-1)|B_{2n+2}|-(-1)^{n+1}\right)2^{2n+1}}{(2n+2)!}x^{2n-1}\tan x$$
$$<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\sum_{k=3}^{n}\frac{\left(2(2^{2k}-1)|B_{2k}|-(-1)^k\right)2^{2k-1}}{(2k)!}x^{2k-2}+\left(\frac{2}{\pi}\right)^{2n}x^{2n-1}\tan x,$$

where B_n $(n\in\mathbb{N}_0,\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}:=\{1,2,\dots\})$ are the Bernoulli numbers, defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\qquad |t|<2\pi.$$

CONJECTURE 1.2. For $0<x<\pi/2$ and $n\geqslant 1$,

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}<2+\sum_{k=2}^{n}\frac{(k-1)\cdot 2^{2k+1}|B_{2k}|}{(2k)!}x^{2k}+\frac{n\cdot 2^{2n+3}|B_{2(n+1)}|}{(2n+2)!}x^{2n+1}\tan x.$$

Here, and throughout this paper, an empty sum is understood to be zero.

CONJECTURE 1.3. For $0<x<\pi/2$ and $n\geqslant 2$,

$$3+\sum_{k=3}^{n}\left(\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{4k}-(-1)^k\right)\frac{2}{(2k-1)!}x^{2k-2}$$
$$+\left(\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{4(n+1)}-(-1)^{n+1}\right)\frac{2}{(2n+1)!}x^{2n-1}\tan x$$
$$<2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}$$
$$<3+\sum_{k=3}^{n}\left(\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{4k}-(-1)^k\right)\frac{2}{(2k-1)!}x^{2k-2}+\left(\frac{2}{\pi}\right)^{2n}x^{2n-1}\tan x.$$

Recently, Chen and Paris [4] proved Conjecture 1.2. This paper is a continuation of our earlier work [4]. The first aim of the present paper is to prove Conjectures 1.1 and 1.3.

Mortici [11, Theorem 1] presented the following double inequality:

$$2+\left(\frac{8}{45}-\frac{8}{945}x^2\right)x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}$$
$$<2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4\right)x^3\tan x,\qquad 0<x<1. \tag{1.11}$$

By using Maple software, we find that

$$\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2}{x^3\tan x}=\frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4+\frac{8}{467775}x^6+\frac{3184}{638512875}x^8$$

$$+\frac{272}{638512875}x^{10}+\frac{7264}{162820783125}x^{12}+\cdots. \tag{1.12}$$

This fact led us to claim that the upper bound in (1.11) should be the lower bound. The second aim of the present paper is to prove the following inequality:

$$\begin{aligned} &2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4\right)x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\\ <&2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{241920-2688\pi^4+32\pi^6}{945\pi^8}x^4\right)x^3\tan x, \qquad 0<x<\frac{\pi}{2}, \end{aligned} \tag{1.13}$$

where the constants $\frac{16}{14175}$ and $(241920-2688\pi^4+32\pi^6)/(945\pi^8)$ are the best possible.

REMARK 1.1. The inequalities (1.13) are sharper than the inequalities (1.2) and (1.8).

In analogy with (1.13), we here determine the best possible constants $\alpha,\beta,\lambda,\mu,\rho$, and ϱ such that

$$2+\left(\frac{2}{45}-\frac{2}{315}x^2-\alpha x^4\right)x^3\tan x<\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}<2+\left(\frac{2}{45}-\frac{2}{315}x^2-\beta x^4\right)x^3\tan x,$$

$$3+\left(\frac{3}{20}+\frac{1}{280}x^2+\lambda x^4\right)x^3\tan x<2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}<3+\left(\frac{3}{20}+\frac{1}{280}x^2+\mu x^4\right)x^3\tan x$$

and

$$3+\left(\frac{1}{60}-\frac{1}{280}x^2-\rho x^4\right)x^3\tan x<2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}<3+\left(\frac{1}{60}-\frac{1}{280}x^2-\varrho x^4\right)x^3\tan x$$

for $0<x<\pi/2$. This is the last aim of the present paper.

2. A useful lemma

It is well known that

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}, \qquad |x|<\frac{\pi}{2}, \tag{2.1}$$

By using induction, Chen and Qi [5] (see also [24]) proved the following

LEMMA 2.1. *Let $n\geqslant 1$ be an integer. Then for $0<x<\pi/2$, we have*

$$\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}x^{2n}\tan x<\tan x-\sum_{k=1}^{n}\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}x^{2k-1}$$

$$< \left(\frac{2}{\pi}\right)^{2n} x^{2n}\tan x, \tag{2.2}$$

where the the constants

$$\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!} \quad \textit{and} \quad \left(\frac{2}{\pi}\right)^{2n}$$

are the best possible.

3. Main results

THEOREM 3.1. *For $0 < x < \pi/2$ and $n \geqslant 3$, we have*

$$\begin{aligned}
&\left(\frac{(-1)^n 2^{2n+1}}{(2n+2)!} + \frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right) x^{2n-1}\tan x \\
&< \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - \left\{2 + \sum_{k=3}^{n}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}\right\} \\
&< \left(\frac{2}{\pi}\right)^{2n} x^{2n-1}\tan x.
\end{aligned} \tag{3.1}$$

Proof. First of all, we prove the first inequality in (3.1). By using the power series expansions for $\cos x$ and $\tan x$, we have

$$\begin{aligned}
\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &= \frac{1}{2x^2} - \frac{1}{2x^2}\cos(2x) + \frac{\tan x}{x} \\
&= 2 + \sum_{k=3}^{n}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2} + r_n(x),
\end{aligned}$$

where

$$r_n(x) = \sum_{k=n+1}^{\infty}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}.$$

The first inequality in (3.1) is equivalent to

$$\left(\frac{(-1)^n 2^{2n+1}}{(2n+2)!} + \frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right) x^{2n-1}\tan x < r_n(x)$$

for $0 < x < \pi/2$ and $n \geqslant 3$, which can be written by (2.1) as

$$\begin{aligned}
\sum_{k=n+2}^{\infty}\Bigg\{&\left(\frac{(-1)^n 2^{2n+1}}{(2n+2)!} + \frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!} \\
&- \left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)\Bigg\}x^{2k-2} < 0,
\end{aligned}$$

where we note that the term corresponding to $k=n+1$ vanishes.

We claim that for $k \geqslant n+2$,

$$\left(\frac{(-1)^n 2^{2n+1}}{(2n+2)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!} < \frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}. \tag{3.2}$$

It is enough to prove the following inequality:

$$\left(\frac{2^{2n+1}}{(2n+2)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!} < -\frac{2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}, \qquad k \geqslant n+2.$$

Using the following inequality (see [7]):

$$\frac{2}{(2\pi)^{2n}(1-2^{-2n})} < \frac{|B_{2n}|}{(2n)!} < \frac{2}{(2\pi)^{2n}(1-2^{1-2n})}, \qquad n \geqslant 1, \tag{3.3}$$

it suffices to show that for $k \geqslant n+2$,

$$\left(\frac{2^{2n+1}}{(2n+2)!}+\frac{2^{2n+2}(2^{2n+2}-1)2}{(2\pi)^{2n+2}\left(1-2^{1-2(n+1)}\right)}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)2}{(2\pi)^{2k-2n}\left(1-2^{1-2(k-n)}\right)} < -\frac{2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)2}{(2\pi)^{2k}(1-2^{-2k})},$$

which can be rearranged as

$$\left(\frac{2^{2n+1}}{(2n+2)!}+\frac{2(2^{2n+2}-1)}{2^{2n+2}-2}\left(\frac{2}{\pi}\right)^{2n+2}\right)\frac{2^{2k-2n}-1}{2^{2k-2n}-2}\left(\frac{2}{\pi}\right)^{2k-2n}+\frac{2^{2k-2}}{(2k)!} < \left(\frac{2}{\pi}\right)^{2k},$$

$$\left(\frac{2^{2n+1}}{(2n+2)!}+2\left(1+\frac{1}{2^{2n+2}-2}\right)\left(\frac{2}{\pi}\right)^{2n+2}\right)\left(1+\frac{1}{2^{2k-2n}-2}\right)\left(\frac{\pi}{2}\right)^{2n} + \frac{2^{2k-2}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k} < 1, \qquad k \geqslant n+2.$$

Noting that the sequences

$$1+\frac{1}{2^{2k-2n}-2} \quad \text{and} \quad \frac{2^{2k-2}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}$$

are both strictly decreasing for $k \geqslant n+2$, it suffices to show that

$$\left(\frac{2^{2n+1}}{(2n+2)!}+2\left(1+\frac{1}{2^{2n+2}-2}\right)\left(\frac{2}{\pi}\right)^{2n+2}\right)\frac{15}{14}\left(\frac{\pi}{2}\right)^{2n}+\frac{2^{2n+2}}{(2n+4)!}\left(\frac{\pi}{2}\right)^{2n+4} < 1$$

for $n \geqslant 3$, which can be rearranged as

$$\frac{2^{2n+1}}{(2n+2)!}\left(\frac{\pi}{2}\right)^{2n+2}+\frac{1}{2^{2n+1}-1}+\frac{2^{2n+2}}{(2n+4)!}\left(\frac{\pi}{2}\right)^{2n+6}\left(\frac{14}{15}\right)<\frac{14}{15}\left(\frac{\pi}{2}\right)^{2}-2,\ n\geqslant 3.$$

Noting that the sequence

$$a_n:=\frac{2^{2n+1}}{(2n+2)!}\left(\frac{\pi}{2}\right)^{2n+2}+\frac{1}{2^{2n+1}-1}+\frac{2^{2n+2}}{(2n+4)!}\left(\frac{\pi}{2}\right)^{2n+6}\left(\frac{14}{15}\right),\qquad n\geqslant 3$$

is strictly decreasing, we see that

$$a_n\leqslant a_3=\frac{1}{127}+\frac{\pi^8}{80640}+\frac{\pi^{12}}{62208000}<\frac{14}{15}\left(\frac{\pi}{2}\right)^{2}-2$$

holds true for $n \geqslant 3$, since

$$\frac{1}{127}+\frac{\pi^8}{80640}+\frac{\pi^{12}}{62208000}=0.14039705\ldots,\qquad \frac{14}{15}\left(\frac{\pi}{2}\right)^{2}-2=0.30290769\ldots.$$

This proves the claim (3.2). Hence, the first inequality in (3.1) holds for $0<x<\pi/2$ and $n \geqslant 3$.

Secondly, we prove the second inequality in (3.1). We consider two cases.

Case 1. $n=2N+1$ $(N\geqslant 1)$.

It is well known that for $x\neq 0$,

$$\sum_{k=1}^{2N}(-1)^{k-1}\frac{x^{2k-2}}{(2k-2)!}<\cos x<\sum_{k=1}^{2N+1}(-1)^{k-1}\frac{x^{2k-2}}{(2k-2)!}.$$

We then obtain that

$$\left(\frac{\sin x}{x}\right)^2=\frac{1}{2x^2}-\frac{1}{2x^2}\cos(2x)<\sum_{k=1}^{2N+1}(-1)^{k-1}\frac{2^{2k-1}}{(2k)!}x^{2k-2}. \tag{3.4}$$

The choice $n=2N+1$ in (2.2), we obtain from the right-hand inequality of (2.2) that

$$\frac{\tan x}{x}<\sum_{k=1}^{2N+1}\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}x^{2k-2}+\left(\frac{2}{\pi}\right)^{4N+2}x^{4N+1}\tan x. \tag{3.5}$$

Adding these two expressions, we obtain

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\sum_{k=3}^{2N+1}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}$$
$$+\left(\frac{2}{\pi}\right)^{4N+2}x^{4N+1}\tan x.$$

This shows that the second inequality in (3.1) holds for $n=2N+1$.

Case 2. $n=2N$ $(N\geqslant 2)$.

Write

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}=2+\sum_{k=3}^{2N}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}$$
$$+\sum_{k=2N+1}^{\infty}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}.$$

We need to prove

$$\sum_{k=2N+1}^{\infty}\left(\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}<\left(\frac{2}{\pi}\right)^{4N}x^{4N-1}\tan x. \tag{3.6}$$

Noting that (2.1) holds, we can rewrite (3.6) as

$$\sum_{k=2N+1}^{\infty}\left\{\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right.$$
$$\left.-\left(\frac{2}{\pi}\right)^{4N}\frac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!}\right\}x^{2k-2}<0.$$

We claim that for $k\geqslant 2N+1$,

$$\frac{(-1)^{k-1}2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}<\left(\frac{2}{\pi}\right)^{4N}\frac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!}. \tag{3.7}$$

It is enough to prove the following inequality:

$$\frac{2^{2k-1}}{(2k)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}<\left(\frac{2}{\pi}\right)^{4N}\frac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!},\qquad k\geqslant 2N+1. \tag{3.8}$$

Using (3.3), we find that for $k\geqslant 2N+1$,

$$\frac{\dfrac{2^{2k+2}(2^{2k+2}-1)|B_{2k+2}|}{(2k+2)!}}{\dfrac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}}<\frac{\dfrac{2^{2k+2}(2^{2k+2}-1)2}{(2\pi)^{2k+2}\left(1-2^{1-2(k+1)}\right)}}{\dfrac{2^{2k}(2^{2k}-1)2}{(2\pi)^{2k}\left(1-2^{-2k}\right)}}=\frac{2(4^k-2)(4\cdot 4^k-1)}{\pi^2(4^k-1)(2\cdot 4^k-1)}<1 \tag{3.9}$$

and[1]

$$\frac{\dfrac{2^{2k-4N+2}(2^{2k-4N+2}-1)|B_{2k-4N+2}|}{(2k-4N+2)!}}{\dfrac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!}}>\frac{\dfrac{2^{2k-4N+2}(2^{2k-4N+2}-1)2}{(2\pi)^{2k-4N+2}\left(1-2^{-2(k-2N+1)}\right)}}{\dfrac{2^{2k-4N}(2^{2k-4N}-1)2}{(2\pi)^{2k-4N}\left(1-2^{1-2(k-2N)}\right)}}$$

[1] The inequality (3.10) is proved in the appendix.

$$= \frac{16^{k+N+1} - (8 \cdot 256^N + 64^N)4^{k+1} + 8 \cdot 1024^N}{\pi^2 (4^k - 16^N)(4^{k+1} - 16^N)} > 1. \tag{3.10}$$

Hence, the sequence

$$\frac{2^{2k-1}}{(2k)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}$$

is strictly decreasing, and the sequence

$$\left(\frac{2}{\pi}\right)^{2N} \frac{2^{2k-2N}(2^{2k-2N}-1)|B_{2k-2N}|}{(2k-2N)!}$$

is strictly increasing for $k \geqslant 2N+1$. In order to prove (3.8), it suffices to show that for $k \geqslant 2N+1$,

$$\frac{2^{4N+1}}{(4N+2)!} + \frac{2^{4N+2}(2^{4N+2}-1)|B_{4N+2}|}{(4N+2)!} < \left(\frac{2}{\pi}\right)^{4N} \frac{2^2(2^2-1)|B_2|}{2!} = \left(\frac{2}{\pi}\right)^{4N}. \tag{3.11}$$

By (3.3), it suffices to show that

$$\frac{2^{4N+1}}{(4N+2)!} + \frac{2^{4N+2}(2^{4N+2}-1)2}{(2\pi)^{4N+2}\left(1-2^{1-2(2N+1)}\right)} < \left(\frac{2}{\pi}\right)^{4N},$$

$$\frac{2^{4N+1}}{(4N+2)!} + 2\left(1 + \frac{1}{2^{4N+2}-2}\right)\left(\frac{2}{\pi}\right)^{4N+2} < \left(\frac{2}{\pi}\right)^{4N},$$

$$\frac{2^{4N+1}}{(4N+2)!}\left(\frac{\pi}{2}\right)^{4N+2} + \frac{1}{2^{4N+1}-1} < \left(\frac{\pi}{2}\right)^2 - 2, \qquad N \geqslant 2.$$

Noting that the sequence

$$b_N := \frac{2^{4N+1}}{(4N+2)!}\left(\frac{\pi}{2}\right)^{4N+2} + \frac{1}{2^{4N+1}-1}, \qquad N \geqslant 2$$

is strictly decreasing, we see that

$$b_N \leqslant b_2 = \frac{1}{511} + \frac{\pi^{10}}{7257600} < \left(\frac{\pi}{2}\right)^2 - 2$$

holds true for $N \geqslant 2$, since

$$\frac{1}{511} + \frac{\pi^{10}}{7257600} = 0.0148603\ldots, \qquad \left(\frac{\pi}{2}\right)^2 - 2 = 0.4674011\ldots .$$

This proves the claim (3.7). Hence, (3.6) holds, which shows that the second inequality in (3.1) holds for $n = 2N$. Thus, the second inequality in (3.1) holds for $0 < x < \pi/2$ and $n \geqslant 3$. The proof of Theorem 3.1 is complete.

THEOREM 3.2. *For $0<x<\pi/2$ and $n\geqslant 2$, we have*

$$\left(\frac{2(-1)^n}{(2n+1)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)x^{2n-1}\tan x$$
$$<2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}-\left\{3+\sum_{k=3}^{n}\left(\frac{2(-1)^{k-1}}{(2k-1)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}\right\}$$
$$<\left(\frac{2}{\pi}\right)^{2n}x^{2n-1}\tan x. \tag{3.12}$$

Proof. First of all, we prove the first inequality in (3.12). By using the power series expansions for $\sin x$ and $\tan x$, we have

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}=3+\sum_{k=3}^{n}\left(\frac{2(-1)^{k-1}}{(2k-1)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}+R_n(x),$$

where

$$R_n(x)=\sum_{k=n+1}^{\infty}\left(\frac{2(-1)^{k-1}}{(2k-1)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}.$$

The first inequality in (3.12) is equivalent to

$$\left(\frac{2(-1)^n}{(2n+1)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)x^{2n-1}\tan x<R_n(x)$$

for $0<x<\pi/2$ and $n\geqslant 2$, which can be written by (2.1) as

$$\sum_{k=n+2}^{\infty}\left\{\left(\frac{2(-1)^n}{(2n+1)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!}\right.$$
$$\left.-\left(\frac{2(-1)^{k-1}}{(2k-1)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)\right\}x^{2k-2}<0,$$

where we note that the term corresponding to $k=n+1$ vanishes.

We claim that for $k\geqslant n+2$,

$$\left(\frac{2(-1)^n}{(2n+1)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!}$$
$$<\frac{2(-1)^{k-1}}{(2k-1)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}. \tag{3.13}$$

It is enough to prove the following inequality:

$$\left(\frac{2}{(2n+1)!}+\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!}$$

$$< -\frac{2}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}, \qquad k \geqslant n+2.$$

Using (3.3), it suffices to show that for $k \geqslant n+2$,

$$\left(\frac{2}{(2n+1)!} + \frac{2^{2n+2}(2^{2n+2}-1)2}{(2\pi)^{2n+2}\left(1-2^{1-2(n+1)}\right)}\right)\frac{2^{2k-2n}(2^{2k-2n}-1)2}{(2\pi)^{2k-2n}\left(1-2^{1-2(k-n)}\right)}$$
$$< -\frac{2}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)2}{(2\pi)^{2k}\left(1-2^{-2k}\right)},$$

which can be rearranged as

$$\left(\frac{1}{(2n+1)!} + \frac{2^{2n+2}-1}{2^{2n+2}-2}\left(\frac{2}{\pi}\right)^{2n+2}\right)\frac{2(2^{2k-2n}-1)}{2^{2k-2n}-2}\left(\frac{2}{\pi}\right)^{2k-2n} + \frac{1}{(2k-1)!} < \left(\frac{2}{\pi}\right)^{2k},$$
$$\left(\frac{1}{(2n+1)!} + \left(1+\frac{1}{2^{2n+2}-2}\right)\left(\frac{2}{\pi}\right)^{2n+2}\right)2\left(1+\frac{1}{2^{2k-2n}-2}\right)\left(\frac{\pi}{2}\right)^{2n}$$
$$+\frac{1}{(2k-1)!}\left(\frac{\pi}{2}\right)^{2k} < 1, \qquad k \geqslant n+2.$$

Noting that the sequences

$$2\left(1+\frac{1}{2^{2k-2n}-2}\right) \quad \text{and} \quad \frac{1}{(2k-1)!}\left(\frac{\pi}{2}\right)^{2k}$$

are both strictly decreasing for $k \geqslant n+2$, it suffices to show that

$$\left(\frac{1}{(2n+1)!} + \left(1+\frac{1}{2^{2n+2}-2}\right)\left(\frac{2}{\pi}\right)^{2n+2}\right)\frac{15}{7}\left(\frac{\pi}{2}\right)^{2n} + \frac{1}{(2n+3)!}\left(\frac{\pi}{2}\right)^{2n+4} < 1$$

for $n \geqslant 2$, which can be rearranged as

$$\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+2} + \frac{1}{2^{2n+2}-2} + \frac{1}{(2n+3)!}\left(\frac{\pi}{2}\right)^{2n+6}\left(\frac{7}{15}\right) < \frac{7}{15}\left(\frac{\pi}{2}\right)^2 - 1, \ n \geqslant 2.$$

Noting that the sequence

$$x_n := \frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+2} + \frac{1}{2^{2n+2}-2} + \frac{1}{(2n+3)!}\left(\frac{\pi}{2}\right)^{2n+6}\left(\frac{7}{15}\right), \qquad n \geqslant 2$$

is strictly decreasing, we see that

$$x_n \leqslant x_2 = \frac{1}{62} + \frac{\pi^6}{7680} + \frac{\pi^{10}}{11059200} < \frac{7}{15}\left(\frac{\pi}{2}\right)^2 - 1$$

holds true for $n \geqslant 2$, since

$$\frac{1}{62} + \frac{\pi^6}{7680} + \frac{\pi^{10}}{11059200} = 0.1497778\ldots, \qquad \frac{7}{15}\left(\frac{\pi}{2}\right)^2 - 1 = 0.1514538\ldots.$$

This proves the claim (3.13). Hence, the first inequality in (3.12) holds for $0 < x < \pi/2$ and $n \geqslant 2$.

Secondly, we prove the second inequality in (3.12). We consider two cases.

Case 1. $n = 2N+1$ $(N \geqslant 1)$.

It is well known that for $x \neq 0$,

$$\sum_{k=1}^{2N}(-1)^{k-1}\frac{x^{2k-2}}{(2k-1)!} < \frac{\sin x}{x} < \sum_{k=1}^{2N+1}(-1)^{k-1}\frac{x^{2k-2}}{(2k-1)!}. \tag{3.14}$$

From the second inequality in (3.14) and (3.5), we obtain

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \sum_{k=3}^{2N+1}\left(\frac{2(-1)^{k-1}}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2} + \left(\frac{2}{\pi}\right)^{4N+2} x^{4N+1}\tan x.$$

This shows that the second inequality in (3.12) holds for $n = 2N+1$.

Case 2. $n = 2N$ $(N \geqslant 1)$.

Write

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} = 3 + \sum_{k=3}^{2N}\left(\frac{2(-1)^{k-1}}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2} + \sum_{k=2N+1}^{\infty}\left(\frac{2(-1)^{k-1}}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2}.$$

We need to prove

$$\sum_{k=2N+1}^{\infty}\left(\frac{2(-1)^{k-1}}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}\right)x^{2k-2} < \left(\frac{2}{\pi}\right)^{4N} x^{4N-1}\tan x. \tag{3.15}$$

Noting that (2.1) holds, we can rewrite (3.15) as

$$\sum_{k=2N+1}^{\infty}\left\{\frac{2(-1)^{k-1}}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} - \left(\frac{2}{\pi}\right)^{4N}\frac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!}\right\}x^{2k-2} < 0.$$

We claim that for $k \geqslant 2N+1$,

$$\frac{2(-1)^{k-1}}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} < \left(\frac{2}{\pi}\right)^{4N}\frac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!}. \tag{3.16}$$

It is enough to prove the following inequality:

$$\frac{2}{(2k-1)!} + \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} < \left(\frac{2}{\pi}\right)^{4N}\frac{2^{2k-4N}(2^{2k-4N}-1)|B_{2k-4N}|}{(2k-4N)!}, \quad k \geqslant 2N+1. \tag{3.17}$$

By (3.9) and (3.10), we see that the sequence

$$\frac{2}{(2k-1)!}+\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}$$

is strictly decreasing, and the sequence

$$\left(\frac{2}{\pi}\right)^{2N}\frac{2^{2k-2N}(2^{2k-2N}-1)|B_{2k-2N}|}{(2k-2N)!}$$

is strictly increasing for $k \geqslant 2N+1$. In order to prove (3.17), it suffices to show that for $k \geqslant 2N+1$,

$$\frac{2}{(4N+1)!}+\frac{2^{4N+2}(2^{4N+2}-1)|B_{4N+2}|}{(4N+2)!}<\left(\frac{2}{\pi}\right)^{4N}\frac{2^2(2^2-1)|B_2|}{2!}=\left(\frac{2}{\pi}\right)^{4N}. \quad (3.18)$$

By (3.3), it now suffices to show that

$$\frac{2}{(4N+1)!}+\frac{2^{4N+2}(2^{4N+2}-1)2}{(2\pi)^{4N+2}\left(1-2^{1-2(2N+1)}\right)}<\left(\frac{2}{\pi}\right)^{4N},$$

which can be rearranged as

$$\left(\frac{2}{\pi}\right)^{4N+2}\frac{1}{(4N+1)!}+\frac{1}{2^{4N+2}-2}<\frac{\pi^2}{8}-1, \qquad N \geqslant 1.$$

Noting that the sequence

$$y_N := \left(\frac{2}{\pi}\right)^{4N+2}\frac{1}{(4N+1)!}+\frac{1}{2^{4N+2}-2}, \qquad N \geqslant 1$$

is strictly decreasing, we see that

$$y_N \leqslant y_1 = \frac{\pi^4}{48384} < \frac{\pi^2}{8}-1$$

holds true for $N \geqslant 1$, since

$$\frac{\pi^4}{48384}=0.00201325\ldots, \qquad \frac{\pi^2}{8}-1=0.23370055\ldots.$$

This proves the claim (3.16). Hence, (3.15) holds, which shows that the second inequality in (3.12) holds for $n=2N$. Thus, the second inequality in (3.12) holds for $0<x<\pi/2$ and $n \geqslant 2$. The proof of Theorem 3.2 is complete.

THEOREM 3.3. *For $0<x<\pi/2$, we have*

$$2+\left(\frac{8}{45}-\frac{8}{945}x^2+ax^4\right)x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}$$

$$< 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + bx^4\right) x^3 \tan x \qquad (3.19)$$

with the best possible constants

$$a = \frac{16}{14175} = 0.001128\ldots \quad \textit{and} \quad b = \frac{241920 - 2688\pi^4 + 32\pi^6}{945\pi^8} = 0.001209\ldots. \qquad (3.20)$$

Proof. The inequality (3.19) can be written as

$$a < f(x) < b,$$

where

$$f(x) = \frac{1}{x^4}\left(\frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x} - \left(\frac{8}{45} - \frac{8}{945}x^2\right)\right).$$

Direct computations yield

$$\lim_{x\to 0} f(x) = \frac{16}{14175} \quad \text{and} \quad \lim_{x\to\pi/2} f(x) = \frac{241920 - 2688\pi^4 + 32\pi^6}{945\pi^8}.$$

In order to prove (3.19), it suffices to show that $f(x)$ is strictly increasing on $(0, \pi/2)$.

Differentiation yields

$$f'(x) = \frac{g(x)}{945x^{10}\sin^2 x},$$

with

$$\begin{aligned} g(x) &= 6615x^2\sin(2x) - 8505\sin^3 x\cos x - 8505x + 1890x^3 \\ &\quad + 10395x\cos^2 x - 1890x\cos^4 x + 672x^5\sin^2 x - 16x^7\sin^2 x \\ &= 6615x^2\sin(2x) - 8505\left(\frac{1}{4}\sin(2x) - \frac{1}{8}\sin(4x)\right) - 8505x + 1890x^3 \\ &\quad + \frac{10395}{2}x\Big(1+\cos(2x)\Big) - 1890x\left(\frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}\right) \\ &\quad + 336x^5\Big(1-\cos(2x)\Big) - 8x^7\Big(1-\cos(2x)\Big) \\ &= \frac{16}{495}x^{13} + \frac{496}{61425}x^{15} - \frac{64}{26325}x^{17} + \sum_{n=9}^{\infty}(-1)^{n-1}u_n(x), \end{aligned}$$

where

$$u_n(x) = \Big(945n\cdot 2^{2n-1} - 16065\cdot 2^{2n-2} + 16n^7 - 112n^6 + 952n^5$$

$$-1960n^4+889n^3+13727n^2-2172n\Big)\frac{2^{2n}x^{2n+1}}{(2n+1)!}, \qquad n \geqslant 9.$$

Direct computation yields

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{8x^2 p_n}{q_n},$$

where

$$p_n = (1890n-16065)4^n+16n^7+616n^5+1680n^4+889n^3+12810n^2+24309n+11340$$

and

$$q_n = (n+1)(2n+3)\Big((1890n-16065)4^n+64n^7-448n^6+3808n^5 \\ -7840n^4+3556n^3+54908n^2-8688n\Big).$$

Noting that $8(\pi/2)^2 < 20$, we find that for $0<x<\pi/2$ and $n \geqslant 9$,

$$\frac{u_{n+1}(x)}{u_n(x)} < \frac{8(\pi/2)^2 p_n}{q_n} < \frac{20p_n}{q_n} < 1,$$

since[2]

$$q_n - 20p_n > 0 \quad \text{for} \quad n \geqslant 9. \tag{3.21}$$

Therefore, for fixed $x \in (0,\pi/2)$, the sequence $n \mapsto u_n(x)$ is strictly decreasing for $n \geqslant 9$. Hence, for $0<x<\pi/2$,

$$g(x) > \frac{16}{495}x^{13} + \frac{496}{61425}x^{15} - \frac{64}{26325}x^{17} = \frac{16}{495}x^{13} + \frac{16x^{15}(93-28x^2)}{184275} > 0.$$

We then obtain that $f'(x)>0$ for $0<x<\pi/2$. The proof of Theorem 3.3 is complete.

Following the same method used in the proof of Theorem 3.3, we can prove the following theorem.

THEOREM 3.4. *For $0<x<\pi/2$, we have*

$$2+\left(\frac{2}{45}-\frac{2}{315}x^2-\alpha x^4\right)x^3\tan x < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \\ < 2+\left(\frac{2}{45}-\frac{2}{315}x^2-\beta x^4\right)x^3\tan x, \tag{3.22}$$

$$3+\left(\frac{3}{20}+\frac{1}{280}x^2+\lambda x^4\right)x^3\tan x < 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}$$

[2] The inequality (3.21) is proved in the appendix.

$$< 3 + \left(\frac{3}{20} + \frac{1}{280}x^2 + \mu x^4\right) x^3 \tan x \tag{3.23}$$

and

$$3 + \left(\frac{1}{60} - \frac{1}{280}x^2 - \rho x^4\right) x^3 \tan x < 2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x}$$
$$< 3 + \left(\frac{1}{60} - \frac{1}{280}x^2 - \varrho x^4\right) x^3 \tan x, \tag{3.24}$$

with the best possible constants

$$\alpha = \frac{224 - 8\pi^2}{315\pi^4} = 0.004727\ldots, \quad \beta = \frac{4}{1575} = 0.002539\ldots, \tag{3.25}$$
$$\lambda = \frac{23}{33600} = 0.000684\ldots, \quad \mu = \frac{17920 - 168\pi^4 - \pi^6}{70\pi^8} = 0.000894\ldots \tag{3.26}$$

and

$$\rho = \frac{56 - 3\pi^2}{210\pi^4} = 0.0012901\ldots, \quad \varrho = \frac{83}{100800} = 0.0008234\ldots. \tag{3.27}$$

Proof. We only prove inequality (3.24). The proofs of (3.22) and (3.23) are analogous. The inequality (3.24) can be written as

$$\rho > F(x) > \varrho,$$

where

$$F(x) = \frac{1}{x^4}\left(-\frac{2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} - 3}{x^3 \tan x} + \left(\frac{1}{60} - \frac{1}{280}x^2\right)\right).$$

Direct computations yield

$$\lim_{x\to 0} F(x) = \frac{83}{100800} \quad \text{and} \quad \lim_{x\to\pi/2} F(x) = \frac{56 - 3\pi^2}{210\pi^4}.$$

In order to prove (3.24), it suffices to show that $F(x)$ is strictly increasing on $(0, \pi/2)$.

Differentiation yields

$$F'(x) = \frac{G(x)}{420x^8 \sin^3 x},$$

with

$$\begin{aligned} G(x) &= 2520x\sin(2x) + (2520x - 3x^5 + 28x^3)\sin x\cos^2 x \\ &\quad + (3x^5 - 1260x - 28x^3)\sin x + (840x^2 - 8820)\cos x + 840x^2\cos^2 x \\ &\quad + 8820\cos^3 x + 840x^2 \end{aligned}$$

$$= 2520x\sin(2x) + (2520x - 3x^5 + 28x^3)\left(\frac{1}{4}\sin x + \frac{1}{4}\sin(3x)\right)$$
$$+ (3x^5 - 1260x - 28x^3)\sin x + (840x^2 - 8820)\cos x + 420x^2\Big(1 + \cos(2x)\Big)$$
$$+ 8820\left(\frac{1}{4}\cos(3x) + \frac{3}{4}\cos x\right) + 840x^2$$
$$= \sum_{n=6}^{\infty}(-1)^n U_n(x),$$

where

$$U_n(x) = \Big((8n^5 - 40n^4 + 238n^3 - 302n^2 - 33924n + 178605)9^n - (34020n^2 + 187110n)4^n$$
$$-5832n^5 + 29160n^4 - 64638n^3 - 215298n^2 + 222588n - 178605\Big)\frac{x^{2n}}{81\cdot(2n)!}.$$

Direct computation yields

$$\frac{U_{n+1}(x)}{U_n(x)} = \frac{9x^2P_n}{2Q_n},$$

where

$$P_n = (8n^5 + 158n^3 + 252n^2 - 33934n + 144585)9^n - (15120n^2 + 113400n + 98280)4^n$$
$$- 648n^5 - 32508n^2 - 34938n - 23625 - 702n^3$$

and

$$Q_n = (2n+1)(n+1)\Big((8n^5 - 40n^4 + 238n^3 - 302n^2 - 33924n + 178605)9^n$$
$$- (34020n^2 + 187110n)4^n - 5832n^5 + 29160n^4 - 64638n^3$$
$$- 215298n^2 + 222588n - 178605\Big).$$

Noting that $\frac{9}{2}\left(\frac{\pi}{2}\right)^2 < 12$, we find that for $0 < x < \pi/2$ and $n \geqslant 6$,

$$\frac{U_{n+1}(x)}{U_n(x)} < \frac{9(\pi/2)^2P_n}{2Q_n} < \frac{12P_n}{Q_n} < 1,$$

since[3]

$$Q_n - 12P_n > 0 \quad \text{for} \quad n \geqslant 6. \tag{3.28}$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \mapsto U_n(x)$ is strictly decreasing for $n \geqslant 6$. Hence, we have

$$G(x) > 0, \qquad 0 < x < \frac{\pi}{2}.$$

We then obtain that $F'(x) > 0$ for $0 < x < \pi/2$. Hence, the inequality (3.24) holds with the best possible constants given in (3.27). The proof is complete.

[3]The inequality (3.28) is proved in the appendix.

REMARK 3.1. The upper bound in (3.22) is sharper than the upper bound in (1.9). The inequalities (3.23) are sharper than the inequalities (1.10).

REMARK 3.2. Chen and Paris [4] showed that for $0 < x < \pi/2$,

$$3+\theta_1 x^3 \tan x < 2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x} < 3+\theta_2 x^3 \tan x \tag{3.29}$$

with the best possible constants

$$\theta_1 = 0 \quad \text{and} \quad \theta_2 = \frac{1}{60}.$$

The double inequality (3.24) is an improvement on the double inequality (3.29).

Appendix A: Proof of (3.10)

Noting that $\pi^2 < 10$, in order to prove (3.10), it suffices to show that for $k \geqslant 2N+1$,

$$\begin{aligned}
&16^{k+N+1}-(8\cdot 256^N+64^N)4^{k+1}+8\cdot 1024^N-10(4^k-16^N)(4^{k+1}-16^N)\\
=&\Big((4^{2N+2}-40)\cdot 4^k+50\cdot 16^N-32\cdot 256^N-4\cdot 64^N\Big)4^k+(8\cdot 1024^N-10\cdot 256^N)\\
>&0.
\end{aligned} \tag{A.1}$$

We see that for $k \geqslant 2N+1$,

$$\begin{aligned}
&(4^{2N+2}-40)\cdot 4^k+50\cdot 16^N-32\cdot 256^N-4\cdot 64^N\\
>&(4^{2N+2}-40)\cdot 4^{2N+1}+50\cdot 16^N-32\cdot 256^N-4\cdot 64^N\\
=&224\cdot 256^N-590\cdot 16^N-4\cdot 64^N>0
\end{aligned}$$

and

$$8\cdot 1024^N-10\cdot 256^N>0.$$

Hence, (A.1) holds for $k \geqslant 2N+1$.

Appendix B: Proof of (3.21)

$$\begin{aligned}
q_n-20p_n=&\Big(3780n^3-22680n^2-112455n+273105\Big)4^n+128n^9-576n^8+5248n^7\\
&+2016n^6-32984n^5+70476n^4+250052n^3-134916n^2-512244n-226800\\
=&\Big(179550+397845(n-9)+79380(n-9)^2+3780(n-9)^3\Big)4^n\\
&+49648561200+46968464520(n-9)+19975332000(n-9)^2\\
&+5019956996(n-9)^3+822741108(n-9)^4+91303912(n-9)^5\\
&+6864480(n-9)^6+337024(n-9)^7+9792(n-9)^8+128(n-9)^9\\
>&0 \quad \text{for} \quad n \geqslant 9.
\end{aligned}$$

Appendix C: Proof of (3.28)

We now show that for $n \geqslant 6$,

$$\begin{aligned}Q_n - 12P_n = &(16n^7 - 56n^6 + 268n^5 - 70412n^3 + 252112n^2 + 909099n - 1556415)9^n \\ &+70n^4 \cdot 9^n - (68040n^4 + 476280n^3 + 413910n^2 - 1173690n - 1179360)4^n \\ &- (11664n^7 - 40824n^6 + 39852n^5 + 595350n^4 + 256932n^3 - 485352n^2 \\ &- 106029n - 104895) > 0.\end{aligned}$$

It suffices to show that for $n \geqslant 6$,

$$\left(\frac{9}{4}\right)^n > A_n, \tag{C.1}$$

where

$$A_n = \frac{68040n^4 + 476280n^3 + 413910n^2 - 1173690n - 1179360}{16n^7 - 56n^6 + 268n^5 - 70412n^3 + 252112n^2 + 909099n - 1556415},$$

and

$$\begin{aligned}9^n > \frac{1}{70n^4}\Big(&11664n^7 - 40824n^6 + 39852n^5 + 595350n^4 + 256932n^3 - 485352n^2 \\ &- 106029n - 104895\Big).\end{aligned} \tag{C.2}$$

By induction with respect to n, we can prove the inequalities (C.1) and (C.2). Here, we only prove the inequality (C.1). The proof of (C.2) is analogous.

For $n = 6$ in (C.1), we find that

$$\left(\frac{9}{4}\right)^6 = \frac{531441}{4096} = 129.746\ldots \quad \text{and} \quad A_6 = \frac{3138660}{27229} = 115.269\ldots.$$

This shows that (C.1) holds for $n = 6$.

Now we assume that (C.1) holds for some $n \geqslant 6$. Then, for $n \mapsto n+1$ in (C.1), by using the induction hypothesis, we have

$$\left(\frac{9}{4}\right)^{n+1} - A_{n+1} > \frac{9}{4}A_n - A_{n+1} = \frac{2835R_n}{2S_nT_n},$$

where

$$\begin{aligned}R_n = &960n^{11} + 12384n^{10} + 73088n^9 + 256200n^8 - 3508908n^7 - 22121984n^6 + 50474996n^5 \\ &+ 274552068n^4 - 445858781n^3 - 777353865n^2 + 997107984n - 660306024 \\ = &878926761468 + 1894841991720(n-6) + 1695853296525(n-6)^2 \\ &+ 849645117283(n-6)^3 + 268187103036(n-6)^4 + 56595283460(n-6)^5\end{aligned}$$

$$+8234103112(n-6)^6+835076820(n-6)^7+58479432(n-6)^8+2716928(n-6)^9 + 75744(n-6)^{10}+960(n-6)^{11},$$

$$\begin{aligned} S_n &= 16n^7-56n^6+268n^5-70412n^3+252112n^2+909099n-1556415 \\ &= 1715427+679323(n-6)+1087672(n-6)^2+509908(n-6)^3+98760(n-6)^4 \\ &\quad +10348(n-6)^5+616(n-6)^6+16(n-6)^7 \end{aligned}$$

and

$$\begin{aligned} T_n &= 16n^7+56n^6+268n^5+1060n^4-68292n^3+43052n^2+1203203n-465388 \\ &= 4102070+4834979(n-6)+3323012(n-6)^2+1021308(n-6)^3+160300(n-6)^4 \\ &\quad +14380(n-6)^5+728(n-6)^6+16(n-6)^7. \end{aligned}$$

Hence, we have

$$\left(\frac{9}{4}\right)^{n+1} > A_{n+1}.$$

Thus, by the principle of mathematical induction, the inequality (C.1) holds for $n \geqslant 6$.

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Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: chenchaoping@sohu.com

Richard B. Paris
Division of Computing and Mathematics
University of Abertay
Dundee, DD1 1HG, UK
e-mail: R.Paris@abertay.ac.uk