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AN INEQUALITY INVOLVING THE CONSTANT $e$ AND A GENERALIZED CARLEMAN–TYPE INEQUALITY

CHAO-PING CHEN AND RICHARD B. PARIS

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Abstract. In this paper, we establish a double inequality involving the constant $e$. As an application, we give a generalized Carleman-type inequality.

1. Introduction

Let $a_n \geq 0$ for $n \in \mathbb{N} := \{1, 2, \ldots\}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \quad (1.1)$$

The constant $e$ is the best possible. The inequality (1.1) was presented in 1922 in [3] by the Swedish mathematician Torsten Carleman and it is called Carleman’s inequality. Carleman discovered this inequality during his important work on quasi-analytical functions.

Carleman’s inequality (1.1) was generalized by Hardy [12] (see also [13, p. 256]) as follows: If $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n (a^{\lambda_1}_1 a^{\lambda_2}_2 \cdots a^{\lambda_n}_n)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \quad (1.2)$$

Note that inequality (1.2) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In his original paper [12], Hardy himself said that it was Pólya who pointed out this inequality to him. For information about the history of Carleman-type inequalities, please refer to [15, 16, 18, 24].

In [4, 5, 6, 9, 10, 11, 14, 19, 20, 21, 22, 23, 26, 27, 28, 29, 30, 31], some strengthened and generalized results of (1.1) and (1.2) have been given by estimating the weight coefficient $(1 + 1/n)^n$. For example, Mortici and Jang [23] proved that for $0 < x \leq 1$,

$$e \left(1 - \frac{1}{2} x + \frac{11}{24} x^2 - \frac{7}{16} x^3 + \frac{2447}{5760} x^4 - \frac{959}{2304} x^5\right) < (1 + x)^{1/x}$$

$$< e \left(1 - \frac{1}{2} x + \frac{11}{24} x^2 - \frac{7}{16} x^3 + \frac{2447}{5760} x^4\right). \quad (1.3)$$


Keywords and phrases: Carleman’s inequality, weight coefficient.

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According to Pólya’s proof of (1.1) in [25],
\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n a_n, \]  
(1.4)
and then the following strengthened Carleman’s inequality can be derived directly from the right-hand side of (1.3):
\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n} + \frac{11}{24n^2} - \frac{7}{16n^3} + \frac{2447}{5760n^4}\right) a_n. \]  
(1.5)

In this paper, we develop the double inequality (1.3) to produce a general result. As an application, we give a generalized Carleman-type inequality.

2. A double inequality involving the constant \( e \)

Brothers and Knox [2] (see also [17, 7]) derived, without a formula for the general term, the following expansion:
\[ \left(1 + \frac{1}{x}\right)^x = e \left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \cdots \right) \]  
(2.1)
for \( x < -1 \) or \( x \geq 1 \). Chen and Choi [7] gave an explicit formula for successively determining the coefficients. More precisely, these authors proved that
\[ \left(1 + \frac{1}{x}\right)^x \sim e \sum_{j=0}^{\infty} (-1)^j b_j x^{-j} \quad (x \to \infty), \]  
(2.2)
where the coefficients \( b_j \) are given by
\[ b_0 = 1 \quad \text{and} \quad b_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{\left(\frac{1}{2}\right)^{k_1} \left(\frac{1}{3}\right)^{k_2} \cdots \left(\frac{1}{j+1}\right)^{k_j}}{k_1!k_2!\cdots k_j!} \]  
(2.3)
summed over all nonnegative integers \( k_j \) satisfying the equation \( k_1 + 2k_2 + \cdots + jk_j = j \).

A recurrence relation for the coefficients \( b_j \) can be obtained by use of the result given in [8, Lemma 3]. This states that for a function \( A(x) \) with asymptotic expansion \( A(x) \sim \sum_{n=1}^{\infty} \alpha_n x^{-n} \) as \( x \to \infty \), the composition \( B(x) = \exp[A(x)] \) has the expansion \( B(x) \sim \sum_{n=1}^{\infty} \beta_n x^{-n} \) as \( x \to \infty \), where \( \beta_0 = 1 \) and
\[ \beta_n = \frac{1}{n} \sum_{k=1}^{n} k \alpha_k \beta_{n-k} \quad (n \geq 1). \]

From the Maclaurin expansion
\[ \frac{1}{x} \ln(1 + x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^j}{j+1} \quad (-1 < x \leq 1), \]
it therefore follows (upon replacing \( x \) by \( 1/x \)) that the coefficients \( b_j \) in (2.2) are given by the recurrence relation

\[
b_0 = 1 \quad \text{and} \quad b_j = \frac{1}{j} \sum_{k=1}^{j} \frac{k}{k+1} b_{j-k} \quad (j \geq 1).
\]  

(2.4)

Use of (2.4) is easily seen to generate the values

\[
b_1 = \frac{1}{2}, \quad b_2 = \frac{11}{24}, \quad b_3 = \frac{7}{16}, \quad b_4 = \frac{2447}{5760}, \quad b_5 = \frac{959}{2304}, \quad b_6 = \frac{238043}{580608} \ldots,
\]

which are the same coefficients as in (2.1). The representation using a recursive algorithm for the coefficients \( b_j \) is more practical for numerical evaluation than the expression in (2.3).

The above result immediately shows that \( b_j > 0 \) so that (2.2) is an alternating series for positive \( x \). Replacement of \( x \) by \( 1/x \) in (2.1) and (2.2) then enables us to write

\[
(1 + x)^{1/x} = e \sum_{j=0}^{\infty} (-1)^j b_j x^j \quad (1 < x \leq 1).
\]  

(2.5)

We now establish a monotonicity property satisfied by the coefficients \( b_j \).

**Lemma 2.1.** The sequence \( \{b_j\}_{j=0}^{\infty} \) in (2.5) is monotonically decreasing.

**Proof.** By Cauchy’s theorem it follows from (2.5) that

\[
b_j = \frac{(-1)^j}{2\pi i} \oint_C (1+t)^{1/t} \frac{dt}{t^{j+1}},
\]

where \( C \) is a closed loop surrounding \( t = 0 \) described in the positive sense. Define

\[
\Delta_j = b_j - b_{j+1}.
\]

Then

\[
\Delta_j = \frac{(-1)^j}{2\pi i} \oint_C (1+t)^{1/t} \left(1 + \frac{1}{t}\right) \frac{dt}{t^{j+1}} = \frac{(-1)^j}{2\pi i} \oint_C (1+t)^{1+1/t} \frac{dt}{t^{j+2}}.
\]

In the \( t \)-plane there is a branch cut along \((-\infty, -1]\). Now expand \( C \) to be a large circle of radius \( R \) that is indented to pass along the upper and lower sides of the branch cut. The contribution from the large circle tends to zero as \( R \to \infty \). Similarly, the contribution round the branch point \( t = -1 + \rho e^{i\theta}, \ -\pi \leq \theta \leq \pi \) vanishes as \( \rho \to 0 \). Then we have upon putting \( t = xe^{\pm \pi i} \) on the upper and lower sides of the branch cut

\[
\Delta_j = \frac{1}{2\pi i} \int_{-\infty}^{1} (x-1)^{1-1/x} e^{-\pi i/x} \frac{dx}{x^{j+2}} + \frac{1}{2\pi i} \int_{1}^{\infty} (x-1)^{1-1/x} e^{\pi i/x} \frac{dx}{x^{j+2}} = \frac{1}{\pi e} \int_{1}^{\infty} (x-1)^{1-1/x} \sin(\pi/x) \frac{dx}{x^{j+2}}.
\]  

(2.6)
Now on the interval \(x \in [1, \infty)\) the function \(\sin(\pi/x) \geq 0\) so that the integrand in (2.6) is non-negative on \([1, \infty)\). Hence \(\Delta_j > 0\) and the sequence \(\{b_j\}_{j=0}^{\infty}\) is monotonically decreasing. This completes the proof.

**Remark 2.1.** We thank a referee for providing the literature [1]. It was proved in [1, Lemma 1] that

\[
(x + 1) \left[ e - \left( 1 + \frac{1}{x} \right)^x \right] = \frac{e}{2} + \int_0^1 \frac{g(s)}{x + s} \, ds \quad (x > 0),
\]

where

\[
g(s) = \frac{1}{\pi} s^t (1 - s)^{1-s} \sin(\pi s) \quad (0 \leq s \leq 1).
\]

By (2.7), we here give an integral representation of the coefficients \(b_j\) in (2.5), and then use it to prove Lemma 2.1.

Write (2.7) as

\[
\left( 1 + \frac{1}{x} \right)^x = e - \frac{e}{2(x + 1)} - \int_0^1 \frac{g(s)}{(x + 1)(x + s)} \, ds \quad (x > 0).
\]

Replacing \(x\) by \(1/t\) in (2.9) yields, for \(t > 0\),

\[
f(t) := (1 + t)^{1/t} = \frac{e}{2} + \frac{e}{2(t + 1)} - \int_0^1 \frac{g(s)}{s} \left\{ 1 + \frac{s}{(1-s)(t+1)} - \frac{1}{s(1-s)(t+\frac{1}{s})} \right\} \, ds.
\]

Clearly,

\[eb_0 = f(0) = e.\]

Differentiating the expression in (2.10), we find that for \(n \geq 1\),

\[
\frac{(-1)^n f^{(n)}(t)}{n!} = \frac{e}{2(t + 1)^{n+1}} - \int_0^1 \frac{g(s)}{s} \left\{ \frac{s}{(1-s)(t+1)^{n+1}} - \frac{1}{s(1-s)(t+\frac{1}{s})^{n+1}} \right\} \, ds,
\]

we then obtain the following integral representation of the coefficients \(b_j\) in (2.5):

\[
b_n = \frac{(-1)^n f^{(n)}(0)}{n! e} = \frac{1}{2} - \frac{1}{e} \int_0^1 \frac{1 - s^{n-1}}{1 - s} g(s) \, ds
\]

for \(n \geq 1\), and we have

\[
\Delta_j = b_j - b_{j+1} = \frac{1}{e} \int_0^1 s^{j-1} g(s) \, ds > 0 \quad (j \geq 1).
\]

Noting that \(b_0 = 1 > \frac{1}{2} = b_1\) holds, we see that the sequence \(\{b_j\}_{j=0}^{\infty}\) in (2.5) is monotonically decreasing.

In fact, by an elementary change of variable \(x = 1/s\) \((0 \leq s \leq 1)\), we see that (2.6) \(\iff\) (2.11).
From (2.5) and Lemma 2.1 we obtain the following theorem that develops the double inequality (1.3) to produce a general result.

**Theorem 2.1.** For all integers \( m \geq 0 \),

\[
e^{2m+1} \sum_{j=0}^{2m+1} (-1)^j b_j x^j < (1+x)^{1/x} < e^{2m} \sum_{j=0}^{2m} (-1)^j b_j x^j \quad (0 < x \leq 1),
\]

(2.12)

or alternatively

\[
e^{2m+1} \sum_{j=0}^{2m+1} (-1)^j b_j x^j \quad (x \geq 1),
\]

(2.13)

where the coefficients \( b_j \) are given by the recursive relation (2.4).

### 3. A generalized Carleman-type inequality

**Theorem 3.1.** Let \( 0 < \lambda_{n+1} \leq \lambda_n \), \( \Lambda_n = \sum_{m=1}^{n} \lambda_m \) (\( \Lambda_n \geq 1 \)), \( a_n \geq 0 \) (\( n \in \mathbb{N} \)) and \( 0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty \). Then for \( 0 < p \leq 1 \),

\[
\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \quad \leq \quad \frac{e^p}{p} \sum_{n=1}^{\infty} \left( \frac{2m}{\sum_{j=0}^{m} (-1)^j b_j} \right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k (c_k a_k)^p \right)^{(1-p)/p},
\]

(3.1)

where \( b_j \) is given by (2.4), and

\[
c_{\lambda_n}^p = \frac{(\Lambda_{n+1})^{\lambda_n}}{\left(\Lambda_n\right)^{\lambda_{n-1}}}.\]

**Proof.** The following inequality:

\[
\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \quad \leq \quad \frac{1}{p} \sum_{m=1}^{\infty} \left( 1 + \frac{1}{\lambda_m} \right)^{p_{\lambda_m} / \lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left( \sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p},
\]

(3.2)

has been proved in Theorem 2.2 of [11] (see also [21, p. 96]). From (3.2) and the right-hand side of (2.13), we obtain (3.1). The proof is complete.

**Remark 3.1.** In Theorem 2.2 of [11], \( c_{\lambda_n}^p = \frac{(\Lambda_{n+1})^{\lambda_n}}{(\Lambda_n)^{\lambda_{n-1}}} \) should be \( c_{\lambda_n}^p = \frac{(\Lambda_{n+1})^{\lambda_n}}{(\Lambda_n)^{\lambda_{n-1}}} \); see [11, p. 44, line 3]. Likewise, \( c_{\lambda_n}^p = \frac{(\Lambda_{n+1})^{\lambda_n}}{(\Lambda_n)^{\lambda_{n-1}}} \) in Theorem 3.1 of [21] should be \( c_{\lambda_n}^p = \frac{(\Lambda_{n+1})^{\lambda_n}}{(\Lambda_n)^{\lambda_{n-1}}} \); see [21, p. 96, Eq. (9)].
The choice $p = 1$ in (3.1) yields
\[ \sum_{n=1}^{\infty} \lambda_n^{-1} (a_1 \lambda_n^{-1} + a_2 \lambda_n^{-2} + \cdots + a_n \lambda_n^{-n}) < e \sum_{n=1}^{\infty} \left( \sum_{j=0}^{2m} (-1)^j b_j \left( \frac{\lambda_n}{\lambda_j} \right)^j \right) \lambda_n a_n. \] (3.3)

Taking $\lambda_n \equiv 1$ in (3.3) we obtain
\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( \sum_{j=0}^{2m} \frac{(-1)^j b_j}{n^j} \right) a_n. \] (3.4)

When $m = 2$ in (3.4) we recover (1.5).

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