

A (p, ν) -EXTENSION OF SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTION H_C

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ABSTRACT. We obtain a (p, ν) -extension of Srivastava's triple hypergeometric function $H_C(\cdot)$ by employing the extended Beta function $B_{p, \nu}(x, y)$ introduced in Parmar et al. [J. Class. Anal. **11** (2017), 91–106]. We give some of the main properties of this extended function, which include several integral representations, the Mellin transform, a differential formula, recursion formulas and a bounded inequality.

1. Introduction and preliminaries

In the present paper, we employ the following notations:

$$\mathbf{N} := \{1, 2, \dots\}, \quad \mathbf{N}_0 := \mathbf{N} \cup \{0\}, \quad \mathbf{Z}_0^- := \mathbf{Z}^- \cup \{0\},$$

where the symbols \mathbf{N} and \mathbf{Z} denote the set of integer and natural numbers; as usual, the symbols \mathbf{R} and \mathbf{C} denote the set of real and complex numbers, respectively.

Hypergeometric functions of a single variable have a long history and arise in numerous branches of mathematics and physics. The Gauss hypergeometric function is defined for $b_1, b_2 \in \mathbf{C}$, $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$ by

$$(1.1) \quad {}_2F_1 \left(\begin{matrix} b_1, b_2 \\ c_1 \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{(b_1)_n (b_2)_n}{(c_1)_n} \frac{z^n}{n!} \quad (|z| < 1),$$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\dots(a+n-1)$. Extensions of this function to include p numerator parameters b_j ($1 \leq j \leq p$) and q denominator parameters c_j ($1 \leq j \leq q$) also find wide application; see [17]. Triple hypergeometric functions (that is functions of three variables x , y and z) have been introduced and investigated by Srivastava and Karlsson [21, Chapter 3] who provide a table of 205 distinct such functions. In [18, 19], Srivastava introduced the triple hypergeometric functions H_A, H_B and H_C of the second order. It is known that H_C and H_B are

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generalizations of Appell's hypergeometric functions F_1 and F_2 , while H_A is the generalization of both F_1 and F_2 .

In the present study, we shall confine our attention to Srivastava's triple hypergeometric function H_C given by [21, p. 43, 1.5(11) to 1.5(13)] (see also [18] and [20, p. 68])

$$(1.2) \quad H_C(b_1, b_2, b_3; c_1; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{m+k}(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_{m+n+k}} \frac{x^m y^n z^k}{m! n! k!}$$

$$= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

Here $B(\alpha, \beta)$ denotes the classical Beta function defined by [13, (5.12.1)]

$$(1.3) \quad B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt, & (\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & ((\alpha, \beta) \in \mathbf{C} \setminus \mathbf{Z}_0^-). \end{cases}$$

The convergence region for the hypergeometric series $H_C(\cdot)$ is given in [11, p. 243] as $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where α, β, γ satisfy the relation

$$(1.4) \quad \alpha + \beta + \gamma - 2\sqrt{(1-\alpha)(1-\beta)(1-\gamma)} < 2.$$

We shall also find it convenient to introduce an additional parameter a into $H_C(\cdot)$ in the form

$$(1.5) \quad H_C^{(a)}(b_1, b_2, b_3; c_1; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+a+m+k, c_1+a+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

which reduces to (1.2) when $a = 0$.

In 1997, Chaudhry et al. [1, Eq. (1.7)] introduced a p -extension of the Beta function $B(x, y)$ given by

$$B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \quad (\operatorname{Re}(p) > 0).$$

Also, Chaudhry et al. [2] employed this function to extend the Gauss hypergeometric series ${}_2F_1(\cdot)$ and its integral representations. A further extension of the Beta function has been given by Choi et al. [8] in the form

$$B_{p,q}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{p}{t} - \frac{q}{1-t}\right] dt \quad (\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0),$$

which reduces to $B(x, y; p)$ when $p = q$. Recently, Parmar et al. [15] have given a different extension of the Beta function in the form

$$(1.6) \quad B_{p,\nu}(x, y) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt,$$

where $\operatorname{Re}(p) > 0$, $\nu \geq 0$ and $K_\nu(z)$ is the modified Bessel function (sometimes known as the Macdonald function) of order ν . When $\nu = 0$, (1.6) reduces to $B(x, y; p)$, since $K_{\frac{1}{2}}(z) = \sqrt{\pi/(2z)}e^{-z}$.

Many authors have studied integral representations of Srivastava's triple hypergeometric function $H_C(\cdot)$ defined in (1.2); see [3–7]. Our aim in this paper is to introduce a (p, ν) -extension of this function, which we denote by $H_{C,p,\nu}(\cdot)$, based on the extended Beta function in (1.6). The Appell hypergeometric function of two variables defined by

$$F_1(b_1, b_2, b_3; c_1; x, y) = \sum_{m,n=0}^{\infty} \frac{(b_2)_n (b_3)_m B(b_1 + m + n, c_1 - b_1)}{B(b_1, c_1 - b_1)} \frac{x^m y^n}{m! n!}$$

where $(|x| < 1, |y| < 1)$ has been extended by replacement of the numerator Beta function (of the same arguments) with $B(x, y; p)$ in [14] and with $B_{p,\nu}(x, y)$ in [9]. Similar extensions of H_A and H_B have been carried out in [10, 16].

The plan of this paper is as follows. The extended function $H_{C,p,\nu}(\cdot)$ is defined in Section 2 and some integral representations are presented involving the modified Bessel function and the Gauss hypergeometric function ${}_2F_1$. The main properties of $H_{C,p,\nu}(\cdot)$ namely, its Mellin transform, a differential formula, a bounded inequality and recursion formulas are established in Sections 3 to 6. Some concluding remarks are made in Section 7.

2. The (p, ν) -extended Srivastava triple hypergeometric function $H_{C,p,\nu}(\cdot)$

Srivastava introduced the triple hypergeometric function $H_C(\cdot)$, together with its integral representations, in [18] and [20]. Here we consider the following (p, ν) -extension of this function, which we denote by $H_{C,p,\nu}(\cdot)$, based on the extended Beta function $B_{p,\nu}(x, y)$ defined in (1.6). This is given by

$$(2.1) \quad H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!},$$

where the parameters $b_1, b_2, b_3 \in \mathbf{C}$ and $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$. The region of convergence is $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where α, β, γ satisfy (1.4). This definition clearly reduces to the original function when $\nu = 0$.

An integral representation for $H_{C,p,\nu}(\cdot)$ involving the Gauss hypergeometric function ${}_2F_1$ defined in (1.1) can be given. We have

THEOREM 2.1. *The following integral representation of the function $H_{C,p,\nu}(\cdot)$ holds for $\operatorname{Re}(p) > 0$, $\operatorname{Re}(b_j) > 0$ ($j = 1, 2, 3$) and $\operatorname{Re}(c_1 - b_1) > 0$:*

$$(2.2) \quad H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1 - b_1)} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{b_1 - \frac{3}{2}} (1-t)^{c_1 - b_1 - \frac{3}{2}} K_{\nu + \frac{1}{2}} \left(\frac{p}{t(1-t)} \right)$$

$$(2.3) \quad \times (1 - xt)^{-b_2} (1 - zt)^{-b_3} {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \frac{(1-t)y}{(1-xt)(1-zt)}\right) dt,$$

where $|x| < 1$, $|y| < 1$ and $|z| < 1$.

PROOF. The proof of integral representation (2.3) follows by use of the extended beta function (1.6) in (2.1), a change in the order of integration and summation (with uniform convergence of the integral) to find

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1 - b_1)} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{b_1 - \frac{3}{2}} (1-t)^{c_1 - b_1 - \frac{3}{2}} K_{\nu + \frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \\ &\quad \times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1 - b_1)_n} \frac{(xt)^m}{m!} \frac{(y(1-t))^n}{n!} \frac{(zt)^k}{k!} dt. \end{aligned}$$

Making use of the result $(a)_{m+n} = (a)_n (a+n)_m$ we can express the treble sum as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(b_2)_n (b_3)_n}{(c_1 - b_1)_n} \frac{(y(1-t))^n}{n!} \sum_{m=0}^{\infty} (b_2 + n)_m \frac{(xt)^m}{m!} \sum_{k=0}^{\infty} (b_3 + n)_k \frac{(zt)^k}{k!} \\ &= (1 - xt)^{-b_2} (1 - zt)^{b_3} \sum_{n=0}^{\infty} \frac{(b_2)_n (b_3)_n}{(c_1 - b_1)_n} \frac{X^n}{n!}, \quad X = \frac{y(1-t)}{(1-xt)(1-zt)}, \end{aligned}$$

where we have employed the binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a)_n w^n}{n!} = (1-w)^{-a} \quad (a \in \mathbf{C}, |w| < 1)$$

to evaluate the sums over m and k . Identification of the sum over n as a Gauss hypergeometric function by (1.1), then yields (2.3). \square

The following variants of (2.3) can be obtained by making appropriate transformations of the integration variable. We have

$$(2.4) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \int_0^{\infty} \xi^{b_1 - \frac{3}{2}} (1 + \xi)^{b_2 + b_3 - c_1 + 1} \Omega_1^{-b_2} \Omega_2^{-b_3} K_{\nu + \frac{1}{2}}\left(\frac{p}{\sigma_1 \sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where

$$\sigma_1 = \frac{\xi}{1 + \xi}, \quad \sigma_2 = \frac{1}{1 + \xi}, \quad \sigma_3 = \frac{1 + \xi}{\sigma_1 \sigma_2}, \quad \Omega_1 = 1 + (1 - x)\xi, \quad \Omega_2 = 1 + (1 - z)\xi;$$

$$(2.5) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \frac{(\beta - \gamma)^{b_1 - \frac{1}{2}} (\alpha - \gamma)^{c_1 - b_1 - \frac{1}{2}}}{(\beta - \alpha)^{c_1 - b_2 - b_3 - 2}} \times \int_{\alpha}^{\beta} \frac{(\xi - \alpha)^{b_1 - \frac{3}{2}} (\beta - \xi)^{c_1 - b_1 - \frac{3}{2}}}{(\xi - \gamma)^{c_1 - b_2 - b_3 - 1}} \Omega_1^{-b_2} \Omega_2^{-b_3}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where, with $\gamma < \alpha < \beta$,

$$\sigma_1 = \frac{(\beta - \gamma)(\xi - \alpha)}{(\beta - \alpha)(\xi - \gamma)}, \quad \sigma_2 = \frac{(\alpha - \gamma)(\beta - \xi)}{(\beta - \alpha)(\xi - \gamma)}, \quad \sigma_3 = \frac{(\alpha - \gamma)(\beta - \alpha)(\beta - \xi)(\xi - \gamma)}{\Omega_1\Omega_2},$$

$$\Omega_1 = (\beta - \alpha)(\xi - \gamma) - x(\beta - \gamma)(\xi - \alpha), \quad \Omega_2 = (\beta - \alpha)(\xi - \gamma) - z(\beta - \gamma)(\xi - \alpha);$$

$$(2.6) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$$

$$= 2 \int_0^{\pi/2} (\sin^2 \xi)^{b_1-1} (\cos^2 \xi)^{c_1-b_1-1} \Omega_1^{-b_2} \Omega_2^{-b_3}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where

$$\sigma_1 = \sin^2 \xi, \quad \sigma_2 = \cos^2 \xi, \quad \sigma_3 = \frac{\cos^2 \xi}{\Omega_1\Omega_2}, \quad \Omega_1 = 1 - x \sin^2 \xi, \quad \Omega_2 = 1 - z \sin^2 \xi;$$

and

$$(2.7) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$$

$$= (1 + \lambda)^{b_1-\frac{1}{2}} \int_0^1 \frac{\xi^{b_1-\frac{3}{2}} (1 - \xi)^{c_1-b_1-\frac{3}{2}}}{(1 + \lambda\xi)^{c_1-b_2-b_3-1}} \Omega_1^{-b_2} \Omega_2^{-b_3}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where, with $\lambda > -1$,

$$\sigma_1 = \frac{(1 + \lambda)\xi}{1 + \lambda\xi}, \quad \sigma_2 = \frac{1 - \xi}{1 + \lambda\xi}, \quad \sigma_3 = \frac{(1 - \xi)(1 + \lambda\xi)}{\Omega_1\Omega_2},$$

$$\Omega_1 = 1 + \lambda\xi - (1 + \lambda)x\xi, \quad \Omega_2 = 1 + \lambda\xi - (1 + \lambda)z\xi.$$

Integral representations (2.4)–(2.7) can be proved directly by using the following transformations

$$(2.4) : \quad t = \frac{\xi}{1 + \xi}, \quad \frac{dt}{d\xi} = \frac{1}{(1 + \xi)^2}$$

$$(2.5) : \quad t = \frac{(\beta - \gamma)(\xi - \alpha)}{(\beta - \alpha)(\xi - \gamma)}, \quad \frac{dt}{d\xi} = \frac{(\beta - \gamma)(\alpha - \gamma)}{(\beta - \alpha)(\xi - \gamma)^2},$$

$$(2.6) : \quad t = \sin^2 \xi, \quad \frac{dt}{d\xi} = 2 \sin \xi \cos \xi$$

$$(2.7) : \quad t = \frac{(1 + \lambda)\xi}{1 + \lambda\xi}, \quad \frac{dt}{d\xi} = \frac{(1 + \lambda)}{(1 + \lambda\xi)^2},$$

in turn in (2.3) to obtain the right-hand side of each result.

Finally, use of the integral representation [13, p. 388]

$${}_2F_1\left(\begin{matrix} b_1, b_2 \\ c_1 \end{matrix}; z\right) = \frac{1}{B(b_2, c_1 - b_2)} \int_0^1 \frac{t^{b_2-1}(1-t)^{c_1-b_2-1}}{(1-zt)^{b_1}} dt, \quad (|\arg(1-z)| < \pi)$$

for $\operatorname{Re}(c_1) > \operatorname{Re}(b_2) > 0$, shows that

$$\begin{aligned} & H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \sqrt{\frac{2p}{\pi}} \int_0^1 \int_0^1 s^{b_2-1} t^{b_1-\frac{3}{2}} (1-s)^{c_1-b_1-b_2-1} (1-t)^{c_1-b_1-\frac{3}{2}} \\ & \quad \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt) - ys(1-t)\}^{b_3}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) ds dt \end{aligned}$$

provided, in addition, $\operatorname{Re}(c_1 - b_1 - b_2) > 0$.

3. The Mellin transform for $H_{C,p,\nu}(\cdot)$

The Mellin transform of a locally integrable function $f(x)$ on $(0, \infty)$ is given by (see, for example, [12, p. 193, Section 2.1])

$$(3.1) \quad \Phi(s) = \mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x) dx,$$

which defines an analytic function in its strip of analyticity $a < \operatorname{Re}(s) < b$. The inverse Mellin transform of the above function (3.1) is defined by

$$f(x) = \mathcal{M}^{-1}\{\Phi(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Phi(s) ds \quad (a < c < b).$$

THEOREM 3.1. *The following Mellin transform of the extended Srivastava triple hypergeometric function $H_{C,p,\nu}(\cdot)$ holds true:*

$$(3.2) \quad \begin{aligned} \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) &= \int_0^\infty p^{s-1} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) dp, \\ &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z), \end{aligned}$$

where $\operatorname{Re}(s) > \nu > 0$, $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$ and $H_C^{(s)}(\cdot)$ is defined in (1.5).

PROOF. Substituting the extended Srivastava function (2.1) into the integral on the left-hand side of (3.2) and changing the order of integration (by the uniform convergence of the integral), we obtain

$$\begin{aligned} \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) &= \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n B(b_1, c_1+n-b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ & \quad \times \left\{ \int_0^\infty p^{s-1} B_{p,\nu}(b_1+m+k, c_1+n-b_1) dp \right\}. \end{aligned}$$

Use of the extended Beta function (1.6) then shows that

$$\mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s)$$

$$= \sqrt{\frac{2}{\pi}} \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ \times \int_0^1 t^{b_1+m+k-\frac{3}{2}} (1-t)^{c_1+n-b_1-\frac{3}{2}} \left\{ \int_0^{\infty} p^{s-\frac{1}{2}} K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dp \right\} dt.$$

If we apply the result [13, (10.43.19)]

$$\int_0^{\infty} w^{s-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(w) dw = 2^{s-\frac{3}{2}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha+1}{2}\right) \quad (|\operatorname{Re}(\alpha)| < \operatorname{Re}(s))$$

followed by the substitution $w = p/(t(1-t))$, we obtain

$$\mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \\ \times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ \times \left\{ \int_0^1 t^{b_1+m+k+s-1} (1-t)^{c_1+n+s-b_1-1} dt \right\}.$$

Evaluation of the integral in terms of the classical Beta function, then finally yields

$$\Phi(s) = \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \\ \times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1 + s + m + k, c_1 + s + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

Identifying the above sum as $H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z)$ defined in (1.5), we obtain the right-hand side of (3.2). \square

COROLLARY 3.1. *The following inverse Mellin formula for $H_{C,p,\nu}(\cdot)$ holds:*

$$H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) = \mathcal{M}^{-1}\{\Phi(s)\} = \\ \frac{\pi^{-3/2}}{4i} \int_{c-i\infty}^{c+i\infty} \left(\frac{2}{p}\right)^s \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z) ds,$$

where $c > \nu$.

4. A differentiation formula for $H_{C,p,\nu}(\cdot)$

THEOREM 4.1. *The following derivative formula for $H_{C,p,\nu}(\cdot)$ holds:*

$$(4.1) \quad \frac{\partial^{M+N+K}}{\partial x^M \partial y^N \partial z^K} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) = \frac{(b_1)_{M+K} (b_2)_{M+N} (b_3)_{N+K}}{(c_1)_{M+N+K}} \\ \times H_{C,p,\nu}(b_1 + M + K, b_2 + M + N, b_3 + N + K; c_1 + M + N + K; x, y, z),$$

where $M, N, K \in \mathbf{N}_0$.

PROOF. If we differentiate partially the series for $\mathcal{H} \equiv H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$ in (2.1) with respect to x we obtain

$$\frac{\partial \mathcal{H}}{\partial x} = \sum_{m=1}^{\infty} \sum_{n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^{m-1}}{(m-1)!} \frac{y^n}{n!} \frac{z^k}{k!}.$$

Making use of the fact that

$$(4.2) \quad B(b_1, c_1+n-b_1) = \frac{(c_1+n)}{b_1} B(b_1+1, c_1+n-b_1)$$

and $(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n$, we have upon setting $m \rightarrow m+1$

$$(4.3) \quad \begin{aligned} \frac{\partial \mathcal{H}}{\partial x} &= \frac{b_1 b_2}{c_1} \sum_{m,n,k=0}^{\infty} \frac{(b_2+1)_{m+n}(b_3)_{n+k}}{(c_1+1)_n} \frac{B_{p,\nu}(b_1+1+m+k, c_1+n-b_1)}{B(b_1+1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= \frac{b_1 b_2}{c_1} H_{C,p,\nu}(b_1+1, b_2+1, b_3; c_1+1; x, y, z). \end{aligned}$$

Repeated application of (4.3) then yields for $M = 1, 2, \dots$

$$\frac{\partial^M \mathcal{H}}{\partial x^M} = \frac{(b_1)_M (b_2)_M}{(c_1)_M} H_{C,p,\nu}(b_1+M, b_2+M, b_3; c_1+M; x, y, z).$$

A similar reasoning shows that

$$(4.4) \quad \begin{aligned} \frac{\partial^{M+1} \mathcal{H}}{\partial x^M \partial y} &= \frac{(b_1)_M (b_2)_M}{(c_1)_M} \sum_{m,k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_2+M)_{m+n}(b_3)_{n+k}}{(c_1+M)_n} \\ &\quad \times \frac{B_{p,\nu}(b_1+M+m+k, c_1+M+n+k)}{B(b_1+M, c_1+M)} \frac{x^m}{m!} \frac{y^{n-1}}{(n-1)!} \frac{z^k}{k!} \\ &= \frac{(b_1)_M (b_2)_{M+1} b_3}{(c_1)_{M+1}} H_{C,p,\nu}(b_1+M, b_2+M+1, b_3+1; c_1+M+1; x, y, z) \end{aligned}$$

upon putting $n \rightarrow n+1$ and using the property of the Beta function in (1.3). Repeated differentiation of (4.4) N times with respect to y then produces

$$\frac{\partial^{M+N} \mathcal{H}}{\partial x^M \partial y^N} = \frac{(b_1)_M (b_2)_{M+N} (b_3)_N}{(c_1)_{M+N}} H_{C,p,\nu}(b_1+M, b_2+M+N, b_3+N; c_1+M+N; x, y, z).$$

Application of the same procedure (making use of (4.2)) to deal with differentiation with respect to z then yields the result stated in (4.1) \square

5. An upper bound for $H_{C,p,\nu}(\cdot)$

THEOREM 5.1. *Let the parameters $c_1 > 0$, $b_j > 0$ ($1 \leq j \leq 3$) with $c_1 - b_1 > 0$ and the variables x, y, z be complex. Then the following bound for $H_{C,p,\nu}(\cdot)$ holds:*

$$(5.1) \quad \begin{aligned} &|H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)| \\ &< \frac{2^\nu |p|^{\nu+1}}{\sqrt{\pi} (\operatorname{Re}(p))^{2\nu+1}} \Gamma(\nu+1/2) H_C^{(\nu)}(b_1, b_2, b_3; c_1; |x|, |y|, |z|), \end{aligned}$$

where $\operatorname{Re}(p) > 0$, $\nu > 0$ and $H_C^{(\nu)}(\cdot)$ is defined in (1.5)

The integral representation of the extension $H_{B,p,\nu}(\cdot)$ in (2.3) is associated with the modified Bessel function of the second kind, for which we have the following expression [13, (10.32.8)]

$$K_{\nu+\frac{1}{2}}(z) = \frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \int_1^\infty e^{-zt}(t^2-1)^\nu dt, \quad (\nu > -1, \operatorname{Re}(z) > 0).$$

In our problem we have $\nu > 0, \operatorname{Re}(z) > 0$. Further, we let $x = \operatorname{Re}(z)$, so that

$$(5.2) \quad |K_{\nu+\frac{1}{2}}(z)| \leq \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \left| \int_1^\infty e^{-zt}(t^2-1)^\nu dt \right| < \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \int_0^1 t^{2\nu} e^{-xt} dt \\ = \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \frac{\Gamma(2\nu+1, x)}{x^{2\nu+1}},$$

where $\Gamma(a, z)$ is the upper incomplete gamma function [13, (8.2.2)]. We can simplify (5.2) by making use of the simple inequality $\Gamma(2\nu+1, x) < \Gamma(2\nu+1)$ to find

$$(5.3) \quad |K_{\nu+\frac{1}{2}}(z)| < \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \frac{\Gamma(2\nu+1)}{x^{2\nu+1}} = \frac{1}{2} \left(\frac{2|z|}{x^2}\right)^{\nu+\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right),$$

upon use of the duplication formula for the gamma function.

PROOF. : Setting $z = p/(t(1-t))$, where $t \in (0, 1)$ and $\operatorname{Re}(p) > 0$, in (5.3) we obtain

$$\left| K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \right| < \frac{1}{2} \left(\frac{2|p|t(1-t)}{(\operatorname{Re}(p))^2}\right)^{\nu+\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right).$$

We shall assume that the parameters $c_1 > 0, b_j > 0$ ($1 \leq j \leq 3$), with $c_1 - b_1 > 0$. Then, from (2.3),

$$|H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)| \leq \frac{2^\nu |p|^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} (\operatorname{Re}(p))^{2\nu+1}} \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n B(b_1, c_1 - b_1 + n)} \\ \times \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!} \int_0^1 t^{b_1+\nu+m+k-1} (1-t)^{c_1-b_1+\nu+n-1} dt \\ < \frac{2^\nu |p|^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} (\operatorname{Re}(p))^{2\nu+1}} \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \\ \times \frac{B(b_1+\nu+m+k, c_1-b_1+\nu+n)}{B(b_1, c_1-b_1+n)} \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!}$$

which is the result stated in (5.1). \square

6. Recursion formulas for $H_{C,p,\nu}(\cdot)$

In this section, we obtain two recursion formulas for the extended Srivastava function $H_{C,p,\nu}(\cdot)$. The first formula gives recursions with respect to the numerator parameters b_2 and b_3 , and the second a recursion with respect to the denominator parameter c_1 .

THEOREM 6.1. *The following recursions for $H_{C,p,\nu}(\cdot)$ with respect to the numerator parameters b_2 and b_3 hold:*

$$(6.1) \quad H_{C,p,\nu}(b_1, b_2 + 1, b_3; c_1; x, y, z) = H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ + \frac{xb_1}{c_1} H_{C,p,\nu}(b_1 + 1, b_2 + 1, b_3; c_1 + 1; x, y, z) \\ + \frac{yb_3}{c_1} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z),$$

$$(6.2) \quad H_{C,p,\nu}(b_1, b_2, b_3 + 1; c_1; x, y, z) = H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ + \frac{yb_2}{c_1} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z) \\ + \frac{zb_1}{c_1} H_{C,p,\nu}(b_1 + 1, b_2, b_3 + 1; c_1 + 1; x, y, z).$$

PROOF. From (2.1) and the result $(b_2 + 1)_{m+n} = (b_2)_{m+n}(1 + m/b_2 + n/b_2)$, we obtain

$$(6.3) \quad H_{C,p,\nu}(b_1, b_2 + 1, b_3; c_1; x, y, z) \\ = \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ = H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ + \frac{x}{b_2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^{m-1} y^n z^k}{(m-1)! n! k!} \\ + \frac{y}{b_2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^{n-1} z^k}{m! (n-1)! k!}.$$

Consider the first sum in (6.3) which we denote by S . Put $m \rightarrow m + 1$ and use the identity $(a)_{n+1} = a(a+1)_n$ to find

$$S = \frac{x}{b_2} \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n+1} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + 1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ = x \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + 1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

Using (4.2), we then obtain

$$(6.4) \quad S = \frac{xb_1}{c_1} \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n} (b_3)_{n+k}}{(c_1 + 1)_n} \frac{B_{p,\nu}(b_1 + 1 + m + k, c_1 + n - b_1)}{B(b_1 + 1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ = \frac{xb_1}{c_1} H_{C,p,\nu}(b_1 + 1, b_2 + 1, b_3; c_1 + 1; x, y, z).$$

Proceeding in a similar manner for the second series in (6.3) with $n \rightarrow n + 1$, we find that this sum can be expressed as

$$(6.5) \quad \frac{yb_3}{c_1} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z).$$

Combination of (6.4) and (6.5) with (6.3) then produces the result stated in (6.1). The proof of (6.2) can be established in a similar manner. \square

COROLLARY 6.1. *From (6.1) and (6.2) the following recursions hold:*

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2 + N, b_3; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{xb_1}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1 + 1, b_2 + \ell, b_3; c_1 + 1; x, y, z) \\ &+ \frac{yb_3}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1, b_2 + \ell, b_3 + 1; c_1 + 1; x, y, z), \end{aligned}$$

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3 + N; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{yb_2}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1, b_2 + 1, b_3 + \ell; c_1 + 1; x, y, z) \\ &+ \frac{zb_1}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1 + 1, b_2, b_3 + \ell; c_1 + 1; x, y, z) \end{aligned}$$

for positive integer N .

THEOREM 6.2. *The following 3-term recursion for $H_{C,p,\nu}(\cdot)$ with respect to the denominator parameter c_1 holds:*

$$(6.6) \quad \begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1 + 1; x, y, z) \\ &+ \frac{yb_2b_3}{c_1(c_1 + 1)} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 2; x, y, z). \end{aligned}$$

PROOF. Consider the case when c_1 is reduced by 1, namely

$$H \equiv H_{C,p,\nu}(b_1, b_2, b_3; c_1 - 1; x, y, z)$$

and use $(c_1 - 1)_n = (c_1)_n / \{1 + \frac{n}{c_1 - 1}\}$. Then

$$\begin{aligned} H &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1 - 1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \left(1 + \frac{n}{c_1 - 1}\right) \frac{x^m y^n z^k}{m! n! k!} \\ &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \end{aligned}$$

$$+ \frac{y}{c_1 - 1} \sum_{m=1}^{\infty} \sum_{n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^{n-1}}{(n-1)!} \frac{z^k}{k!}.$$

Putting $n \rightarrow n+1$ in the above sum, we obtain

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1 - 1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{yb_2b_3}{c_1(c_1 - 1)} \sum_{m,n,k=0}^{\infty} \frac{(b_2+1)_{m+n} (b_3+1)_{n+k}}{(c_1+1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+1+n-b_1)}{B(b_1, c_1+1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) + \frac{yb_2b_3}{c_1(c_1 - 1)} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z). \end{aligned}$$

Replacement of c_1 by $c_1 + 1$ then yields the result stated in (6.6). \square

7. Concluding remarks

We have introduced the (p, ν) -extension of Srivastava's triple hypergeometric function given by $H_{C,p,\nu}(\cdot)$ in (2.1). We have given some integral representations of this function that involve the modified Bessel function of the second kind and a Gauss hypergeometric function. We have also established some properties of the function $H_{C,p,\nu}(\cdot)$, namely the Mellin transform, a differential formula, a bounded inequality and some recursion relations.

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