

## REFINED ASYMPTOTICS OF THE RIEMANN-SIEGEL THETA FUNCTION

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**Abstract.** The Riemann-Siegel theta function  $\vartheta(t)$  is examined for  $t \rightarrow +\infty$ . Use of the refined asymptotic expansion for  $\log \Gamma(z)$  shows that the expansion of  $\vartheta(t)$  contains an infinite sequence of increasingly subdominant exponential terms, each multiplied by an asymptotic series involving inverse powers of  $\pi t$ . Numerical examples are given to detect and confirm the presence of the first three of these exponentials.

**Keywords.** Riemann-Siegel theta function, Gamma function, asymptotic expansion, Stokes phenomenon

**MSC.** 11M06, 30E05, 33B15, 34E05, 41A60

### 1. Introduction

The Riemann-Siegel theta function  $\vartheta(t)$ , which arises when the Riemann zeta function  $\zeta(s)$  is expressed as the real-valued function  $Z(t) := \exp(i\vartheta(t))\zeta(\frac{1}{2}+it)$  on the critical line  $\Re(s) = \frac{1}{2}$ , is defined for real  $t$  by

$$\vartheta(t) = \arg \Gamma\left(\frac{1}{2}it + \frac{1}{4}\right) - \frac{1}{2}t \log \pi. \quad (1.1)$$

This function satisfies  $\vartheta(0) = 0$  and is clearly an odd function of  $t$ , so that it is sufficient to consider  $t > 0$ . The well-known asymptotic expansion for  $\vartheta(t)$  is given by [4, p. xiv]

$$\vartheta(t) \sim \frac{1}{2}t(\log(t/2\pi) - 1) - \frac{\pi}{8} + \sum_{k=1}^{\infty} \frac{(1 - 2^{1-2k})|B_{2k}|}{4k(2k-1)t^{2k-1}} + \frac{1}{2} \arctan(e^{-\pi t}) \quad (1.2)$$

valid as  $t \rightarrow +\infty$ , where  $B_{2k}$  denote the Bernoulli numbers.

Recently, Brent [2] employed an alternative representation of  $\vartheta(t)$ , obtained by application of the reflection and duplication formulas for the gamma function, in the form

$$\vartheta(t) = \frac{1}{2} \arg \Gamma\left(\frac{1}{2} + it\right) - \frac{t}{2} \log 2\pi - \frac{\pi}{8} + \frac{1}{2} \arctan(e^{-\pi t}).$$

By means of an improved error bound for the asymptotic expansion of  $\log \Gamma(z)$  valid in  $\Re(z) \geq 0$ ,  $z \neq 0$ , he was able to derive a rigorous error bound for the expansion in (1.2) valid for all  $K \geq 1$  and  $t > 0$  when the sum on the right-hand side of (1.2) is truncated after  $K$  terms.

The inclusion of the exponentially small term  $\frac{1}{2} \arctan(e^{-\pi t})$  was shown to improve the attainable accuracy of this bound. This term, which may be expressed for  $t > 0$  as a series of decreasing exponentials since

$$\arctan(e^{-\pi t}) = e^{-\pi t} - \frac{1}{3}e^{-3\pi t} + \frac{1}{5}e^{-5\pi t} - \frac{1}{7}e^{-7\pi t} + \dots, \quad (1.3)$$

represents, as we shall show below, the first in a sequence of small exponentials present in the asymptotic expansion of  $\vartheta(t)$ . A further application of the duplication formula enables  $\vartheta(t)$  to be written in terms of  $\Gamma(it)$  and  $\Gamma(2it)$  as [3]

$$\begin{aligned} \vartheta(t) &= \frac{1}{2} \Im \log \{\Gamma(2it)/\Gamma(it)\} - \frac{1}{2}t \log 8\pi - \frac{\pi}{8} + \frac{1}{2} \arctan(e^{-\pi t}) \\ &= \frac{1}{2}t(\log(t/2\pi) - 1) + \frac{1}{2} \Im \{\Omega(2it) - \Omega(it)\} - \frac{\pi}{8} + \frac{1}{2} \arctan(e^{-\pi t}), \end{aligned} \quad (1.4)$$

where

$$\log \Gamma(it) = \left(it - \frac{1}{2}\right) \log it - it + \frac{1}{2} \log 2\pi + \Omega(it), \quad (1.5)$$

with  $\Omega(it)$  denoting the slowly varying part of  $\Gamma(it)$ .

The refined asymptotic expansion of  $\log \Gamma(z)$ , first discussed in [8] and subsequently in [1] and described in detail in [7, §6.4], can then be brought to bear on the asymptotic expansion of  $\vartheta(t)$  for large  $t > 0$ . It was shown [1, 8] that the imaginary  $z$ -axis is a Stokes line for  $\log \Gamma(z)$ , where in the neighbourhood of  $\arg z = \pm \frac{1}{2}\pi$  an infinite number of increasingly subdominant exponential terms switch on as  $|\arg z|$  increases. In this paper we shall exploit this theory to reveal the exponentially small contributions present in the expansion of  $\vartheta(t)$ . To achieve this, and to present a reasonably self-contained account, we reproduce the essential features of the refined asymptotics of  $\log \Gamma(z)$  when  $\arg z = \frac{1}{2}\pi$  in Section 2. The details of our calculation for  $\vartheta(t)$  are presented in Section 3 with a numerical verification given in Section 4.

## 2. The refined asymptotics of $\log \Gamma(z)$ when $\arg z = \frac{1}{2}\pi$

The slowly varying part of the gamma function  $\Omega(z)$  defined by  $\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z+\Omega(z)}$  is shown in [7, p. 282] to be given by the Mellin-Barnes integral

$$\Omega(z) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Gamma(s)}{\sin \pi s} (2\pi z)^{-s} \zeta(1+s) \sin \frac{1}{2} \pi s ds$$

$$= \sum_{k \geq 1} \frac{1}{k} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Gamma(s)}{\sin \pi s} (2\pi z)^{-s} \sin \frac{1}{2} \pi s ds$$

where  $|\arg z| \leq \pi - \delta$ ,  $\delta > 0$  and  $0 < c < 1$ . Displacement of the integration path to the right over the simple poles of the integrand at  $s = 1, 3, \dots, 2n_k - 1$ , where  $\{n_k\}_{k \geq 1}$  is (for the moment) an arbitrary set of positive integers, and use of Cauchy's theorem then shows that, provided  $|\arg z| \leq \pi - \delta$ ,

$$\Omega(z) = \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \sum_{r=0}^{n_k-1} \frac{(-)^r (2r)!}{(2\pi k z)^{2r+1}} + \mathcal{R}(z; n_k) \quad (2.1)$$

With  $L$  denoting the path parallel to the imaginary axis  $\Re(s) = -c + 2n_k + 1$ ,  $0 < c < 1$ , the remainder term is given by (see [7, p. 283])

$$\begin{aligned} \mathcal{R}(z; n_k) &= \sum_{k \geq 1} \frac{1}{k} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)}{\sin \pi s} (2\pi k z)^{-s} \sin \frac{1}{2} \pi s ds \\ &= \sum_{k \geq 1} \frac{1}{k} \left\{ e^{2\pi i k z} T_{2n_k+1}(2\pi i k z) - e^{-2\pi i k z} T_{2n_k+1}(-2\pi i k z) \right\}, \end{aligned} \quad (2.2)$$

where  $T_\nu(z)$  is the so-called *terminant function* defined as a multiple of the incomplete gamma function  $\Gamma(a, z)$  by [7, p. 242]

$$T_\nu(z) = \frac{e^{\pi \nu i} \Gamma(\nu)}{2\pi i} \Gamma(1 - \nu, z) = \frac{e^{\pi i \nu} z^{-\nu} e^{-z}}{4\pi} \int_{-c-\infty i}^{-c+\infty i} \frac{\Gamma(s + \nu)}{\sin \pi s} z^{-s} ds \quad (2.3)$$

with the integral being valid when  $|\arg z| < \frac{3}{2}\pi$  and  $0 < c < 1$ . Thus we have the expansion

$$\Omega(z) = \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \sum_{r=0}^{n_k-1} \frac{(-)^r (2r)!}{(2\pi k z)^{2r+1}} + \mathcal{R}(z; n_k) \quad (2.4)$$

in the sector  $|\arg z| \leq \pi - \delta$ . It is important to stress that the infinite sum on the right-hand side of (2.2) is absolutely convergent (since  $\Gamma(1 - \nu, z) = O(z^{-\nu} e^{-z})$  for fixed  $\nu$  and large  $|z|$  in  $|\arg z| \leq \frac{3}{2}\pi - \delta$  [6, (8.11.2)]) and that (2.4) is consequently *exact*.

Then, when  $z = it$  ( $t > 0$ ), we obtain from (2.4)

$$\Omega(it) = \frac{-i}{\pi} \sum_{k \geq 1} \frac{1}{k} \sum_{r=0}^{n_k-1} \frac{(2r)!}{(2\pi k t)^{2r+1}} + \mathcal{R}(it; n_k), \quad (2.5)$$

where

$$\mathcal{R}(it; n_k) = \sum_{k \geq 1} \frac{1}{k} \left\{ e^{-2\pi k t} T_{2n_k+1}(2\pi k t e^{\pi i}) - e^{2\pi k t} T_{2n_k+1}(2\pi k t) \right\} \quad (2.6)$$

Two important features of the expansion (2.5) are that: (i) the Stirling series appearing in the standard expansion of  $\log \Gamma(it)$  has been decomposed into a  $k$ -sequence of component asymptotic series with scale  $2\pi kt$ , each associated with its own arbitrary truncation index  $n_k$ , and (ii) the order of the terminant functions depends on  $n_k$ . It is these two features that permit each finite series in (2.5) to be optimally truncated for large  $t$  near its least term; that is, when

$$n_k = N_k \sim \pi kt. \quad (2.7)$$

When  $n_k$  is chosen to be the optimal truncation index  $N_k$  for the  $k$ th series, the order and argument of the corresponding terminant functions are approximately equal. To proceed further we now require the asymptotic expansions of  $T_\nu(x)$  and  $T_\nu(xe^{\pi i})$  when  $\nu \sim x$  as  $x \rightarrow +\infty$ .

### 2.1. Asymptotic expansions of $T_\nu(x)$ and $T_\nu(xe^{\pi i})$ .

The asymptotic expansion of  $T_\nu(z)$  for large  $|\nu|$  and  $|z|$ , when  $|\nu| \sim |z|$ , has been discussed in detail by Olver [5]. His analysis was based on the Laplace integral representation

$$T_\nu(z) = \frac{e^{(\pi-\theta)\nu i} e^{-z}}{2\pi i} \int_0^\infty e^{-|z|t} \frac{t^{\nu-1}}{1+te^{-i\theta}} dt \quad (z = xe^{i\theta}, x > 0, |\theta| < \pi). \quad (2.8)$$

Letting

$$\nu = x + a, \quad \nu > 0, \quad (2.9)$$

where  $a$  is bounded, we find that when  $\theta = 0$  the above integral is associated with a saddle point at  $t = 1$  and may be written as

$$T_\nu(x) = \frac{e^{\pi \nu i} e^{-2x}}{2\pi i} \int_{-\infty}^\infty e^{-\frac{1}{2}xw^2} \frac{t^{a-1}}{1+t} \frac{dt}{dw} dw, \quad \frac{1}{2}w^2 = t - \log t - 1.$$

The branch of  $w(t)$  is chosen such that  $w \sim t - 1$  as  $t \rightarrow 1$  and reversion of the  $w$ - $t$  mapping yields

$$t = 1 + w + \frac{1}{3}w^2 + \frac{1}{36}w^3 - \frac{1}{270}w^4 + \frac{1}{4320}w^5 + \dots,$$

from which we can compute, with the help of *Mathematica*, the series expansion

$$\frac{t^{a-1}}{1+t} \frac{dt}{dw} = \frac{1}{2} \sum_{s=0}^{\infty} \mathbf{A}_s(a) w^s.$$

Substitution of this expansion in the integral for  $T_\nu(x)$  then yields [5, §5]

$$T_\nu(x) \sim \frac{-ie^{\pi i \nu} e^{-2x}}{2\sqrt{2\pi x}} \sum_{s=0}^{\infty} \mathbf{A}_{2s}(a) \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{1}{2}x\right)^{-s} \quad (x \rightarrow +\infty), \quad (2.10)$$

where the first five even coefficients are

$$\begin{aligned}
 \mathbf{A}_0(a) &= 1, & \mathbf{A}_2(a) &= \frac{1}{6}(2 - 6a + 3a^2), \\
 \mathbf{A}_4(a) &= \frac{1}{24 \cdot 6^2}(-11 - 120a + 300a^2 - 192a^3 + 36a^4), \\
 \mathbf{A}_6(a) &= \frac{1}{300 \cdot 6^4}(-587 + 3510a + 9765a^2 - 26280a^3 + 18900a^4 - 5400a^5 + 540a^6), \quad (2.11) \\
 \mathbf{A}_8(a) &= \frac{1}{5600 \cdot 6^6}(120341 - 44592a - 521736a^2 - 722880a^3 + 2336040a^4 - 1826496a^5 \\
 &\quad + 635040a^6 - 103680a^7 + 6480a^8), \\
 \mathbf{A}_{10}(a) &= \frac{1}{58800 \cdot 6^8}(-4266772 - 14047026a + 18368889a^2 + 19144272a^3 + 23661792a^4 \\
 &\quad - 88817904a^5 + 73929240a^6 - 28921536a^7 + 5987520a^8 - 635040a^9 + 27216a^{10}).
 \end{aligned}$$

As  $\theta \rightarrow \pi$ , the integral (2.8) has a saddle almost coincident with the pole at  $t = e^{i(\theta-\pi)}$  and may be continued analytically to the interval  $\pi \leq \theta < 2\pi$  provided the integration path is deformed to pass over the pole. When  $\theta = \pi$ , it is found that [5, §5]

$$T_\nu(xe^{\pi i}) \sim \frac{1}{2} - \frac{i}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \mathbf{B}_{2s}(a) \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{1}{2}x\right)^{-s}. \quad (2.12)$$

The coefficients  $\mathbf{B}_s(a)$  are computed from the expansion

$$\frac{t^{a-1}}{1-t} \frac{dt}{dw} = -\frac{1}{w} + \sum_{k=0}^{\infty} \mathbf{B}_s(a) w^s,$$

where it is found that the first five even-order coefficients are

$$\begin{aligned}
 \mathbf{B}_0(a) &= \frac{2}{3} - a, & \hat{\mathbf{B}}_2 &= \frac{1}{15 \cdot 6^2}(46 - 225a + 270a^2 - 90a^3), \\
 \mathbf{B}_4(a) &= \frac{1}{70 \cdot 6^4}(230 - 3969a + 11340a^2 - 11760a^3 + 5040a^4 - 756a^5), \\
 \mathbf{B}_6(a) &= \frac{1}{350 \cdot 6^6}(-3626 - 17781a + 183330a^2 - 397530a^3 + 370440a^4 - 170100a^5 \\
 &\quad + 37800a^6 - 3240a^7), \\
 \mathbf{B}_8(a) &= \frac{1}{231000 \cdot 6^8}(-4032746 + 43924815a + 88280280a^2 - 743046480a^3 \quad (2.13) \\
 &\quad + 1353607200a^4 - 1160830440a^5 + 541870560a^6 - 141134400a^7 \\
 &\quad + 19245600a^8 - 1069200a^9), \\
 \mathbf{B}_{10}(a) &= \frac{1}{7007000 \cdot 6^{10}}(502522570 + 1850358861a - 12222960750a^2 - 12894191310a^3 \\
 &\quad + 103403860560a^4 - 167009778936a^5 + 133973920080a^6 - 62315613360a^7 \\
 &\quad + 17552414880a^8 - 2951348400a^9 + 272432160a^{10} - 10614240a^{11}).
 \end{aligned}$$

## 2.2. The asymptotic expansion of $\Im \Omega(it)$ .

From (2.6) and the expansions in (2.10) and (2.12), with the truncation indices  $n_k$  chosen to be the optimal indices  $N_k$  defined in (2.7), we obtain

$$\Im \mathcal{R}(it; N_k) \sim - \sum_{k \geq 1} \frac{e^{-2\pi kt}}{2\pi \sqrt{k^3 t}} \sum_{s=0}^{\infty} \mathbf{C}_s(a_k) (\pi kt)^{-s}, \quad (2.14)$$

where, from (2.9) with  $\nu = 2N_k + 1$  and  $x = 2\pi kt$ ,

$$a_k := 2(N_k - \pi kt) + 1. \quad (2.15)$$

The coefficients  $\mathbf{C}_s(a)$  are defined by

$$\mathbf{C}_s(a) := \left( \frac{1}{2} \mathbf{A}_{2s}(a) + \mathbf{B}_{2s}(a) \right) \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{1}{2})},$$

and from the values given in (2.11) and (2.13) we have

$$\begin{aligned} \mathbf{C}_0(a) &= \frac{7}{6} - a, & \mathbf{C}_1(a) &= \frac{1}{30 \cdot 6^2} (136 - 495a + 405a^2 - 90a^3), \\ \mathbf{C}_2(a) &= \frac{1}{1120 \cdot 6^3} (-695 - 20538a + 54180a^2 - 43680a^3 + 13860a^4 - 1512a^5), \\ \mathbf{C}_3(a) &= \frac{1}{1120 \cdot 6^5} (-15953 + 55929a + 388395a^2 - 949410a^3 + 767340a^4 \\ &\quad - 283500a^5 + 49140a^6 - 3240a^7), \\ \mathbf{C}_4(a) &= \frac{1}{70400 \cdot 6^8} (170640893 + 21630510a - 598217400a^2 - 2559569760a^3 + 6176233800a^4 \\ &\quad - 5034007440a^5 + 2026775520a^6 - 436233600a^7 + 48114000a^8 - 2138400a^9), \\ \mathbf{C}_5(a) &= \frac{1}{25025 \cdot 2^{11} \cdot 6^7} (-8649703370 - 28280511909a + 27178306155a^2 + 28170272130a^3 \\ &\quad + 154158404400a^4 - 357524183016a^5 + 292552139880a^6 - 124352308080a^7 \\ &\quad + 30395645280a^8 - 4313509200a^9 + 330810480a^{10} - 10614240a^{11}). \end{aligned} \quad (2.16)$$

Thus we finally obtain the desired expansion

$$\Im \Omega(it) \sim - \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \sum_{r=0}^{N_k-1} \frac{(2r)!}{(2\pi kt)^{2r+1}} - \frac{1}{2\pi t^{1/2}} \sum_{k \geq 1} \frac{e^{-2\pi kt}}{k^{3/2}} \sum_{s=0}^{\infty} \mathbf{C}_s(a_k) (\pi kt)^{-s} \quad (2.17)$$

as  $t \rightarrow +\infty$ , where  $N_k \sim \pi kt$  is the optimal truncation index of the  $k$ th series in the first sum and the parameters  $a_k$  appearing in the coefficients  $\mathbf{C}_s(a_k)$  are determined by (2.15).

## 3. Asymptotic expansion of $\vartheta(t)$

From (1.4) we require the large- $t$  expansion of the quantity

$$\Upsilon := \frac{1}{2} \Im \{ \Omega(2it) - \Omega(it) \}.$$

The series part of  $\Upsilon$  is, from (2.17), given by

$$\frac{1}{2\pi} \sum_{k \geq 1} \frac{1}{k} \left\{ \sum_{r=0}^{N_k-1} \frac{(2r)!}{(2\pi kt)^{2r+1}} - \sum_{r=0}^{2N_k-1} \frac{(2r)!}{(4\pi kt)^{2r+1}} \right\}. \quad (3.1)$$

The summation over  $k$  may be carried out by a straightforward regrouping of the terms in the above absolutely convergent double series as described in [1] and [7, p. 286]:

$$\begin{aligned} \sum_{k \geq 1} \sum_{r=0}^{N_k-1} \frac{\alpha_r}{k^{2r+2}} &= \sum_{r=0}^{N_1-1} \alpha_r \sum_{k \geq 1} \frac{1}{k^{2r+2}} + \sum_{r=N_1}^{N_2-1} \alpha_r \sum_{k \geq 2} \frac{1}{k^{2r+2}} + \sum_{r=N_2}^{N_3-1} \alpha_r \sum_{k \geq 3} \frac{1}{k^{2r+2}} + \dots \\ &= \sum_{m=1}^{\infty} \sum_{r=N_{m-1}}^{N_m-1} \alpha_r \zeta(2r+2, m), \end{aligned}$$

where  $\alpha_r := (2r)!/(2\pi t)^{2r+1}$  for the first series in (3.1),  $N_0 \equiv 0$  and  $\zeta(s, m) = \sum_{k=0}^{\infty} (k+m)^{-s}$  is the Hurwitz zeta function. This yields for the series contribution to  $\Upsilon$  the result

$$\frac{1}{2\pi} \sum_{m=1}^{\infty} \left\{ \sum_{r=N_{m-1}}^{N_m-1} \frac{(2r)! \zeta(2r+2, m)}{(2\pi t)^{2r+1}} - \sum_{r=2N_{m-1}}^{2N_m-1} \frac{(2r)! \zeta(2r+2, m)}{(4\pi t)^{2r+1}} \right\},$$

The part of  $\Upsilon$  involving terminant functions becomes, from (2.17),

$$\begin{aligned} \frac{1}{4\pi t^{1/2}} \left\{ \sum_{k \geq 1} \frac{e^{-2\pi kt}}{k^{3/2}} \sum_{s=0}^{\infty} \mathbf{C}_s(a_k) (\pi kt)^{-s} - 2^{-1/2} \sum_{k \geq 1} \frac{e^{-4\pi kt}}{k^{3/2}} \sum_{s=0}^{\infty} \mathbf{C}_s(a_{2k}) (2\pi kt)^{-s} \right\} \\ = \frac{1}{4\pi t^{1/2}} \sum_{k \geq 1} \frac{(-)^{k-1} e^{-2\pi kt}}{k^{3/2}} \sum_{s=0}^{\infty} \mathbf{C}_s(a_k) (\pi kt)^{-s}. \end{aligned}$$

Thus we obtain the following expansion theorem which is the main result of the paper:

**THEOREM 3.1.** *The following expansion for the Riemann-Siegel theta function holds:*

$$\begin{aligned} \vartheta(t) &\sim \frac{1}{2} t (\log(t/2\pi) - 1) - \frac{\pi}{8} + \frac{1}{2} \arctan(e^{-\pi t}) \\ &+ \frac{1}{2\pi} \sum_{m=1}^{\infty} \left\{ \sum_{r=N_{m-1}}^{N_m-1} \frac{(2r)! \zeta(2r+2, m)}{(2\pi t)^{2r+1}} - \sum_{r=2N_{m-1}}^{2N_m-1} \frac{(2r)! \zeta(2r+2, m)}{(4\pi t)^{2r+1}} \right\} \\ &+ \frac{1}{4\pi t^{1/2}} \sum_{k \geq 1} \frac{(-)^{k-1} e^{-2\pi kt}}{k^{3/2}} \sum_{s=0}^{\infty} \mathbf{C}_s(a_k) (\pi kt)^{-s} \end{aligned} \quad (3.2)$$

as  $t \rightarrow +\infty$ , where  $N_k \sim \pi kt$  is the optimal truncation index of the  $k$ th asymptotic series, with  $N_0 \equiv 0$ . The first few coefficients  $\mathbf{C}_s(a_k)$  are given in (2.16) with the parameters  $a_k$  specified in (2.15).

Thus we have established that there is an infinite number of exponentially small contributions to the asymptotics of  $\vartheta(t)$  for large  $t > 0$  of the form  $\exp[-2\pi kt]$ ,  $k = 1, 2, \dots$ , each multiplied by an asymptotic series in inverse powers of  $\pi kt$ . In addition, there is an infinite number of exponentials of the form  $\exp[-(2k-1)\pi t]$  resulting from the expansion of  $\arctan(e^{-\pi t})$  in (1.4).

We observe that when the series in  $\Upsilon$  are both truncated at the index  $n$  for all  $k \geq 1$ , the sum appearing in (3.1) becomes

$$\begin{aligned} \frac{1}{2\pi} \sum_{k \geq 1} \frac{1}{k} \left\{ \sum_{r=0}^{n-1} \frac{(2r)!}{(2\pi kt)^{2r+1}} - \sum_{r=0}^{n-1} \frac{(2r)!}{(4\pi kt)^{2r+1}} \right\} &= \frac{1}{2\pi} \sum_{k \geq 1} \frac{1}{k} \sum_{r=0}^{n-1} \frac{(2r)!(1-2^{-2r-1})}{(2\pi kt)^{2r+1}} \\ &= \frac{1}{2\pi} \sum_{r=1}^n \frac{(2r-2)!(1-2^{1-2r})}{(2\pi t)^{2r-1}} \zeta(2r) = \sum_{r=1}^n \frac{(1-2^{1-2r})|B_{2r}|}{4r(2r-1)t^{2r-1}} \end{aligned}$$

upon use of the identity connecting the zeta function of argument  $2r$  to the Bernoulli numbers  $B_{2r}$  given by  $\zeta(2r) = (2\pi)^{2r}|B_{2r}|/(2(2r)!)$ . Thus, when the truncation index is chosen to be fixed and independent of  $k$  the above sum reduces to the (truncated) standard series given in (1.2).

### 3.1. Numerical results

In order to verify the presence of the exponentially small terms in our asymptotic formula for  $\vartheta(t)$ , we first subtract off the main expansion terms by defining

$$\Theta(t) := \vartheta(t) - \frac{1}{2}t(\log(t/2\pi) - 1) + \frac{\pi}{8} - \frac{1}{2} \arctan(e^{-\pi t}).$$

The (truncated) asymptotic series and the subdominant exponential terms appearing in (3.2) are written as

$$\begin{aligned} H_p(t) &:= \frac{1}{2\pi} \sum_{m=1}^p \left\{ \sum_{r=N_{m-1}}^{N_m-1} \frac{(2r)!\zeta(2r+2, m)}{(2\pi t)^{2r+1}} - \sum_{r=2N_{m-1}}^{2N_m-1} \frac{(2r)!\zeta(2r+2, m)}{(4\pi t)^{2r+1}} \right\}, \\ E_k(t) &:= \frac{(-)^{k-1} e^{-2\pi kt}}{4\pi k (kt)^{1/2}} \sum_{s=0}^n \mathbf{C}_s(a_k) (\pi kt)^{-s}, \end{aligned}$$

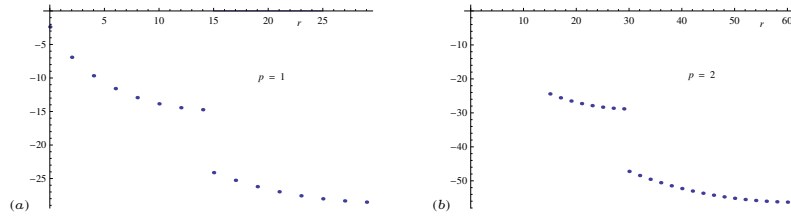
where  $p, k \geq 1$  and  $n \geq 0$  are integers.

To detect the presence of the  $K$ th exponential  $E_K(t)$  it is necessary to ‘‘peel off’’ from  $\vartheta(t)$  all exponentials corresponding to  $k < K$  and all larger terms in the asymptotic series  $H_p(t)$ . In the case  $p = 1$ , we define

$$F_1(t) := \Theta(t) - H_1(t).$$



In Fig. 1(a) the terms of  $H_1(t)$  are plotted against ordinal number  $r$  when  $t = 5$ ; the jump arises because the first sum involves  $N_1$  terms while the second sum involves  $2N_1$  terms. It is seen that this sum contains terms that are much smaller than the first exponential since  $e^{-10\pi} \simeq 2 \times 10^{-14}$ . Thus we expect that  $F_1(t) \sim E_1(t)$  as a leading approximation. In Table 1 we show the computed value of  $F_1(t)$  compared with the value of the first subdominant exponential  $E_1(t)$  for different truncation index  $0 \leq n \leq 5$ . It is seen that there is excellent agreement.



**Figure 1.** Magnitude of the terms (on a  $\log_{10}$  scale) in  $H_p(t)$  against ordinal number  $r$  when  $t = 5$  and truncation indices  $N_1 = 15$ ,  $N_2 = 30$ : (a)  $p = 1$  and (b)  $p = 2$ .

**Table 1.** Values of the first subdominant exponential  $E_1(t)$  as a function of truncation index  $n$  for different  $t$ -values. The last line shows the value of  $F_1(t)$ .

$n$	$t = 5, N_1 = 15$ $E_1(t)$	$t = 8, N_1 = 25$ $E_1(t)$	$t = 10, N_1 = 31$ $E_1(t)$
0	$1.27911882757028 \times 10^{-15}$	$1.79827918803825 \times 10^{-24}$	$1.29604390329845 \times 10^{-29}$
1	$1.29905367126033 \times 10^{-15}$	$1.79141907764499 \times 10^{-24}$	$1.29848420144033 \times 10^{-29}$
2	$1.29933526581188 \times 10^{-15}$	$1.79141504299701 \times 10^{-24}$	$1.29846890774446 \times 10^{-29}$
3	$1.29933821202001 \times 10^{-15}$	$1.79141590147063 \times 10^{-24}$	$1.29846891012541 \times 10^{-29}$
4	$1.29933826830296 \times 10^{-15}$	$1.79141588761045 \times 10^{-24}$	$1.29846891181012 \times 10^{-29}$
5	$1.29933826928455 \times 10^{-15}$	$1.79141588757452 \times 10^{-24}$	$1.29846891177340 \times 10^{-29}$
$F_1(t)$	$1.29933826977440 \times 10^{-15}$	$1.79141588758449 \times 10^{-24}$	$1.29846891177366 \times 10^{-29}$

To demonstrate the presence of the next two exponentials, we set

$$F_2(t) := \Theta(t) - H_2(t) - E_1(t)$$

and

$$F_3(t) := \Theta(t) - H_3(t) - E_1(t) + E_2(t).$$

In Fig. 1(b) the terms in  $H_2(t)$  associated with  $m = 2$  are shown, from which it is seen that they decrease below the value  $e^{-20\pi} \simeq 5 \times 10^{-28}$ ; a similar conclusion arises (not

shown) for the terms in  $H_3(t)$  associated with  $m = 3$ , where  $e^{-30\pi} \sim 1 \times 10^{-41}$ . Thus we expect the leading behaviours to satisfy  $F_2(t) \sim -E_2(t)$  and  $F_3(t) \sim E_3(t)$ .

A difficulty arises in the computation of the exponential series appearing in  $F_1(t)$  and  $F_2(t)$ , since these need to be evaluated at optimal truncation (or at least as accurate as the following exponential series). With only the coefficients  $C_s(a)$  with  $s \leq 5$  it is found that this is insufficient to achieve optimal truncation. To circumvent this problem, we computed the exponential terms in  $F_2(t)$  and  $F_3(t)$  from the terminant function representation in (2.6) using the definition in terms of incomplete gamma functions in (2.3). The results are presented in Table 2, which clearly confirm the expansion in Theorem 1.

**Table 2.** Values of the second and third subdominant exponentials as a function of truncation index  $n$  when  $t = 5$  with truncation indices  $N_1 = 15$ ,  $N_2 = 30$ ,  $N_3 = 45$ . The last line shows the value of  $F_j(t)$  ( $j = 2, 3$ ).

$n$	$E_2(t)$	$E_3(t)$
0	$-1.9459871830 \times 10^{-29}$	$3.5416763558 \times 10^{-43}$
1	$-2.0025102726 \times 10^{-29}$	$3.6851106466 \times 10^{-43}$
2	$-2.0043436323 \times 10^{-29}$	$3.6914163246 \times 10^{-43}$
3	$-2.0044003285 \times 10^{-29}$	$3.6916753673 \times 10^{-43}$
4	$-2.0044020224 \times 10^{-29}$	$3.6916854260 \times 10^{-43}$
5	$-2.0044020723 \times 10^{-29}$	$3.6916858015 \times 10^{-43}$
$F_j(t)$	$-2.0044020737 \times 10^{-29}$	$3.6916858157 \times 10^{-43}$

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