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An Alternative Summation Formula for the Hypergeometric Series ${}_{p+2}F_{p+1}(1)$ with Integral Parameter Differences

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Abstract

We obtain a summation formula for the hypergeometric series of unit argument ${}_{p+2}F_{p+1}$ with p pairs of numeratorial and denominatorial parameters differing by positive integers. This summation formula is an alternative representation to that presented by Miller and Paris [*Rocky Mountain J. Math.* **43** (1) (2013), 291–327] and involves a p-fold sum with coefficients that contain only ratios of Pochhammer symbols. The case when one of the parameters is a negative integer is also considered.

Mathematics Subject Classification: 33C15, 33C20, 33C50

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1. Introduction

The generalised hypergeometric function ${}_pF_q(z)$ is defined for complex parameters and argument by the series [7, p. 40]

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}. $$

When $q = p$ this series converges for $|z| < \infty$, but when $q = p - 1$ convergence occurs when $|z| < 1$. However, when only one of the numeratorial parameters

a_j is a negative integer or zero, then the series converges since it is then a polynomial in z of degree $-a_j$. In the special case $z = 1$, the series converges provided

$$\Re\Big\{\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\Big\} > 0.$$

The Pochhammer symbol, or ascending factorial, $(a)_k$ is defined by $(a)_0 = 1$ and for $k \geq 1$ by $(a)_k = a(a+1)\ldots(a+k-1)$. We shall adopt the convention of writing the finite sequence of parameters $(a_1, \ldots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k := (a_1)_k (a_2)_k \ldots (a_p)_k.$$

The classical formula for the ${}_2F_1(1)$ series is the Gauss summation theorem

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\Re(c-a-b) > 0). \tag{1.1}$$

The case when $a = -n$, where n is a nonnegative integer, results in (1.1) being a terminating series. From (1.1) we have

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}\,\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n},$$

which is known as the Chu-Vandermonde summation theorem. When a pair of numeratorial and denominatorial parameters differs by a positive integer m, Karlsson [1] deduced the extension of the Gauss summation formula in the form

$$ {}_3F_2\left(\begin{matrix} a, b, f+m \\ c, \quad f \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k (a)_k (b)_k}{(f)_k (1+a+b-c)_k}, \tag{1.2}$$

where here and in the sequel we assume that the parameters of all the hypergeometric series are such that they converge and the summation formulas for them make sense. When $a = -n$, (1.2) yields

$$ {}_3F_2\left(\begin{matrix} -n, b, f+m \\ c, \quad f \end{matrix}\,\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n} \sum_{k=0}^{\min\{m,n\}} \binom{m}{k} \frac{(-1)^k (-n)_k (b)_k}{(f)_k (1+b-c-n)_k}, \tag{1.3}$$

The special case $m = 1$ was shown by Miller [2] to be expressible also in the form

$$ {}_3F_2\left(\begin{matrix} -n, b, f+1 \\ c, \quad f \end{matrix}\,\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n} \frac{(\xi+1)_n}{(\xi)_n}, \tag{1.4}$$

where $\xi = f(c-b-1)/(f-b)$ provided $b \neq f$, $c \neq b+1$.

A generalisation of (1.2) has been obtained in [3] and rederived in [4, Theorem 6] in a somewhat simpler form, namely for $p \geq 1$

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} a, b, (f_p+m_p) \\ c, \quad (f_p) \end{matrix} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \alpha_k \frac{(a)_k(b)_k}{(1+a+b-c)_k}, \tag{1.5} $$

where $m \equiv m_1+\cdots+m_p$ and (m_p) is a non-empty sequence of positive integers. In [5], it is shown that the coefficients α_k are given by the set of polynomials

$$ \alpha_k = \frac{1}{k!}\,{}_{p+1}F_p\left(\begin{matrix} -k, (f_p+m_p) \\ (f_p) \end{matrix} \middle| 1\right) \qquad (0 \leq k \leq m). $$

When $c = b+1$, then (1.5) reduces to the Karlsson-Minton summation formula [1, 6] given by

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} a, b, \ (f_p+m_p) \\ 1+b, \quad (f_p) \end{matrix} \middle| 1\right) = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(f_1-b)_{m_1}\dots(f_p-b)_{m_p}}{(f_1)_{m_1}\dots(f_p)_{m_p}} $$

provided $-\Re(a) > m-1$.

In addition, the generalisation of (1.3) was shown in [4, Theorem 1] to be

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} -n, b, (f_p+m_p) \\ c, \quad (f_p) \end{matrix} \middle| 1\right) = \frac{(c-b-m)_n}{(c)_n} \frac{((\xi_m+1))_n}{((\xi_m))_n}, \tag{1.6} $$

where $m = m_1+\cdots+m_p$, $b \neq f_j$ $(1 \leq j \leq p)$ and (ξ_m) are the non-vanishing zeros of the associated polynomial $Q_m(t)$ of degree m given by [4, (2.4)]

$$ Q_m(t) = \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} S(j,k)(b)_k(t)_k(c-b-m-t)_{m-k}, $$

where $S(j,k)$ is the Stirling number of the second kind and the σ_j $(0 \leq j \leq m)$ are determined by the generating relation

$$ (f_1+x)_{m_1}\dots(f_p+x)_{m_p} = \sum_{j=0}^{m} \sigma_{m-j} x^j. $$

The case $p = m = 1$ reduces to Miller's formula (1.4).

The form (1.5) has the attractive feature of involving a single summation, though at the expense of the coefficients α_k possessing a representation in terms of a terminating, lower-order hypergeometric series also of unit argument. Similarly, although the form (1.6) is elegant, the difficulty in its use arises from the determination of the zeros (ξ_m) for general values of p. Our aim in this note is to present alternative forms of (1.5) and (1.6) that involve p summations but with coefficients involving only ratios of Pochhammer symbols.

2. A rederivation of Karlsson's formula

We first present an alternative derivation of Karlsson's summation formula (1.2), which will be used in the following section. We make use of the inverse factorial expansion

$$\frac{(a+m)_r}{(a)_r}=\sum_{k=0}^{m}\binom{m}{k}\frac{r!}{(a)_k(r-k)!} \tag{2.1}$$

for non-negative integers m and r. Then we have (provided $\Re(c-a-b-m)>0$)

$${}_3F_2\left(\begin{matrix}a,b,f+m\\ c,\quad f\end{matrix}\,\middle|\,1\right)=\sum_{r\geq 0}\frac{(a)_r(b)_r}{(c)_r}\,\frac{(f+m)_r}{(f)_r r!}$$

$$=\sum_{r\geq 0}\frac{(a)_r(b)_r}{(c)_r}\sum_{k=0}^{m}\binom{m}{k}\frac{1}{(f)_k(r-k)!}=\sum_{k=0}^{m}\binom{m}{k}\frac{1}{(f)_k}\sum_{r\geq 0}\frac{(a)_r(b)_r}{(c)_r(r-k)!}.$$

In the second sum we make the change $r\to r+k$ and use the identity

$$(a)_{r+k}=(a)_k(a+k)_r$$

to obtain

$${}_3F_2\left(\begin{matrix}a,b,f+m\\ c,\quad f\end{matrix}\,\middle|\,1\right)=\sum_{k=0}^{m}\binom{m}{k}\frac{(a)_k(b)_k}{(c)_k(f)_k}\sum_{r\geq 0}\frac{(a+k)_r(b+k)_r}{(c+k)_r r!}.$$

The sum over r equals

$${}_2F_1\left(\begin{matrix}a+k,b+k\\ c+k\end{matrix}\,\middle|\,1\right)=\frac{\Gamma(c+k)\Gamma(c-a-b-k)}{\Gamma(c-a)\Gamma(c-b)}$$

provided $\Re(c-a-b-k)>0$, $0\leq k\leq m$. Then after a little simplification using

$$\Gamma(c-a-b-k)=\frac{(-)^k\Gamma(c-a-b)}{(1+a+b-c)_k},$$

we finally obtain

$${}_3F_2\left(\begin{matrix}a,b,f+m\\ c,\quad f\end{matrix}\,\middle|\,1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\sum_{k=0}^{m}\binom{m}{k}\frac{(-)^k(a)_k(b)_k}{(f)_k(1+a+b-c)_k},$$

which is Karlsson's result stated in (1.2).

3. The alternative summation formula

Consider first the case $p = 2$ in (1.5). We assume that the convergence condition $\Re(c-a-b-m_1-m_2) > 0$ is satisfied. Then making use of the expansion (2.1), we have

$$
{}_4F_3\left(\begin{matrix} a, b, f_1+m_1, f_2+m_2 \\ c, \quad f_1, \quad f_2 \end{matrix}\,\middle|\, 1\right) = \sum_{r\geq 0} \frac{(a)_r(b)_r(f_1+m_1)_r}{(c)_r(f_1)_r} \sum_{j_2=0}^{m_2} \binom{m_2}{j_2} \frac{1}{(f_2)_{j_2}(r-j_2)!}
$$

$$
= \sum_{j_2=0}^{m_2} \binom{m_2}{j_2} \frac{(a)_{j_2}(b)_{j_2}(f_1+m_1)_{j_2}}{(c)_{j_2}(f_1)_{j_2}(f_2)_{j_2}}\, {}_3F_2\left(\begin{matrix} a+j_2, b+j_2, f_1+m_1+j_2 \\ c+j_2, f_1+j_2 \end{matrix}\,\middle|\, 1\right)
$$

by the same arguments as employed in Section 2. The ${}_3F_2(1)$ series can be summed by Karlsson's formula (1.2) to yield

$$
{}_3F_2\left(\begin{matrix} a+j_2, b+j_2, f_1+m_1+j_2 \\ c+j_2, f_1+j_2 \end{matrix}\,\middle|\, 1\right)
$$

$$
= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \frac{(-)^{j_2}(c)_{j_2}}{(1+a+b-c)_{j_2}} \sum_{j_1=0}^{m_1} \binom{m_1}{j_1} \frac{(-)^{j_1}(a+j_2)_{j_1}(b+j_2)_{j_1}}{(f_1+j_2)_{j_1}(1+a+b-c+j_2)_{j_1}}.
$$

Hence we obtain

$$
{}_4F_3\left(\begin{matrix} a, b, f_1+m_1, f_2+m_2 \\ c, \quad f_1, \quad f_2 \end{matrix}\,\middle|\, 1\right)
$$

$$
= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{j_2=0}^{m_2} \binom{m_2}{j_2} \frac{(-)^{j_2}(a)_{j_2}(b)_{j_2}(f_1+m_1)_{j_2}}{(f_1)_{j_2}(f_2)_{j_2}(1+a+b-c)_{j_2}}
$$

$$
\times \sum_{j_1=0}^{m_1} \binom{m_1}{j_1} \frac{(-)^{j_1}(a+j_2)_{j_1}(b+j_2)_{j_1}}{(f_1+j_2)_{j_1}(1+a+b-c+j_2)_{j_1}}.
$$

If we define the quantities

$$
A_1^{(2)} = \frac{(a+j_2)_{j_1}(b+j_2)_{j_1}}{(f_1+j_2)_{j_1}(1+a+b-c+j_2)_{j_1}}, \quad A_2^{(2)} = \frac{(a)_{j_2}(b)_{j_2}(f_1+m_1)_{j_2}}{(f_1)_{j_2}(f_2)_{j_2}(1+a+b-c)_{j_2}}
$$

this last result can be written more compactly as

$$
{}_4F_3\left(\begin{matrix} a, b, f_1+m_1, f_2+m_2 \\ c, \quad f_1, \quad f_2 \end{matrix}\,\middle|\, 1\right)
$$

$$
= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} (-)^{j_1+j_2} \binom{m_1}{j_1} \binom{m_2}{j_2} A_1^{(2)} A_2^{(2)}
$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}(-)^{j_1+j_2}\binom{m_1}{j_1}\binom{m_2}{j_2}\frac{(a)_{j_1+j_2}(b)_{j_1+j_2}}{(1+a+b-c)_{j_1+j_2}}\,\frac{(f_1+m_1)_{j_2}}{(f_1)_{j_1+j_2}(f_2)_{j_2}} \tag{3.1}$$

after some routine algebra to simplify the product $A_1^{(2)}A_2^{(2)}$.

Proceeding in the same manner for the case $p=3$, we obtain

$$ {}_5F_4\left(\begin{matrix} a,b,(f_3+m_3)\\ c,\quad (f_3)\end{matrix}\,\middle|\,1\right)$$

$$= \sum_{j_3=0}^{m_3}\binom{m_3}{j_3}\frac{(a)_{j_3}(b)_{j_3}((f_2+m_2))_{j_3}}{(c)_{j_3}((f_3))_{j_3}}\,{}_4F_3\left(\begin{matrix} a+j_3,b+j_3,((f_2+m_2+j_3))\\ c+j_3,((f_2+j_3))\end{matrix}\,\middle|\,1\right).$$

Use of (3.1) to sum the ${}_4F_3(1)$ series then leads to

$$ {}_5F_4\left(\begin{matrix} a,b,(f_3+m_3)\\ c,\quad (f_3)\end{matrix}\,\middle|\,1\right)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\sum_{j_3=0}^{m_3}(-)^{j_1+j_2+j_3}\binom{m_1}{j_1}\binom{m_2}{j_2}\binom{m_3}{j_3}A_1^{(3)}A_2^{(3)}A_3^{(3)},$$

where

$$\begin{aligned} A_1^{(3)} &= \frac{(a+j_2+j_3)_{j_1}(b+j_2+j_3)_{j_1}}{(f_1+j_2+j_3)_{j_1}(1+a+b-c+j_2+j_3)_{j_1}},\\ A_2^{(3)} &= \frac{(a+j_3)_{j_2}(b+j_3)_{j_2}(f_1+m_1+j_3)_{j_2}}{((f_2+j_3))_{j_2}(1+a+b-c+j_3)_{j_2}},\\ A_3^{(3)} &= \frac{(a)_{j_3}(b)_{j_3}((f_2+m_2))_{j_3}}{((f_3))_{j_3}(1+a+b-c)_{j_3}}. \end{aligned} \tag{3.2}$$

Some routine algebra shows that

$$A_1^{(3)}A_2^{(3)}A_3^{(3)} = \frac{(a)_{j_1+j_2+j_3}(b)_{j_1+j_2+j_3}}{(1+a+b-c)_{j_1+j_2+j_3}}\,\frac{(f_1+m_1)_{j_2+j_3}(f_2+m_2)_{j_3}}{(f_1)_{j_1+j_2+j_3}(f_2)_{j_2+j_3}(f_3)_{j_3}}.$$

whence we obtain

$$ {}_5F_4\left(\begin{matrix} a,b,(f_3+m_3)\\ c,\quad (f_3)\end{matrix}\,\middle|\,1\right)$$

$$\begin{aligned} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\sum_{j_3=0}^{m_3}(-)^{j_1+j_2+j_3}\binom{m_1}{j_1}\binom{m_2}{j_2}\binom{m_3}{j_3}\\ &\quad\times\frac{(a)_{j_1+j_2+j_3}(b)_{j_1+j_2+j_3}}{(1+a+b-c)_{j_1+j_2+j_3}}\,\frac{(f_1+m_1)_{j_2+j_3}(f_2+m_2)_{j_3}}{(f_1)_{j_1+j_2+j_3}(f_2)_{j_2+j_3}(f_3)_{j_3}}. \end{aligned} \tag{3.3}$$

We remark that in (3.2) and elsewhere we have slightly abused the notation since the factor $((f_2+j_3))_{j_2}$ is to be interpreted as

$$((f_2+j_3))_{j_2} = (f_1+j_3)_{j_2}(f_2+j_3)_{j_2}.$$

This process can be continued in an obvious manner to yield the final result stated in the theorem below.

Theorem 1. *Let the integers J_r be defined by $J_r := j_r + \cdots + j_p$ $(1 \le r \le p)$. Then, provided $\Re(c-a-b-m) > 0$, $m := m_1 + \cdots + m_p$, we have the summation formula*

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} a, b, (f_p+m_p) \\ c, \quad (f_p) \end{matrix} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{j_1=0}^{m_1} \cdots \sum_{j_p=0}^{m_p} (-)^{J_1} \binom{m_1}{j_1} \cdots \binom{m_p}{j_p}$$

$$\times \frac{(a)_{J_1}(b)_{J_1}}{(1+a+b-c)_{J_1}} \frac{(f_1+m_1)_{J_2}(f_2+m_2)_{J_3}\ldots(f_{p-1}+m_{p-1})_{J_p}}{(f_1)_{J_1}(f_2)_{J_2}\ldots(f_p)_{J_p}}. \tag{3.4}$$

The summation formula (3.4) is an alternative version of the formula in (1.5) in which the coefficients are ratios of Pochhammer symbols.

In the particular case $a = -n$, where n is a positive integer, we find from (3.4) the generalisation of the Chu-Vandermonde formula (1.3) in the form:

Theorem 2. *Let the integers J_r be defined by $J_r := j_r + \cdots + j_p$ $(1 \le r \le p)$. Then for positive integer n we have*

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} -n, b, (f_p+m_p) \\ c, \quad (f_p) \end{matrix} \middle| 1\right) = \frac{(c-b)_n}{(c)_n} \sum_{j_1=0}^{m_1^*} \cdots \sum_{j_p=0}^{m_p^*} (-)^{J_1} \binom{m_1}{j_1} \cdots \binom{m_p}{j_p}$$

$$\times \frac{(-n)_{J_1}(b)_{J_1}}{(1+b-c-n)_{J_1}} \frac{(f_1+m_1)_{J_2}(f_2+m_2)_{J_3}\ldots(f_{p-1}+m_{p-1})_{J_p}}{(f_1)_{J_1}(f_2)_{J_2}\ldots(f_p)_{J_p}}, \tag{3.5}$$

where $m_j^* := \min\{m_j, n\}$.

The summation formula (3.5) is an alternative version of the formula in (1.6) involving the zeros (ξ_m) of the associated polynomial $Q_m(t)$.

4. Concluding remarks

We have obtained summation formulas for the ${}_{p+2}F_{p+1}(1)$ hypergeometric series defined in (1.5) with p pairs of numeratorial and denominatorial parameters differing by positive integers for general parameter a and when a is a negative integer. These formulas involve a p-fold summation but with coefficients that contain only ratios of Pochhammer symbols. Our summation formulas provide alternative representations to the forms given in [4, Theorems 1, 6] which, in spite of their apparent simplicity, belie the fact that evaluation of the coefficients and the zeros (ξ_m) of the associated polynomial of degree $m = m_1 + \cdots + m_p$ is difficult for general p.

As a final remark, we add that the formulas (3.4) and (3.5) have been verified numerically using *Mathematica*.

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