MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 1 (2022), 71–77
March 2022

research paper
оригинални научни рад

HYPERGEOMETRIC REPRESENTATIONS OF GELFOND'S CONSTANT AND ITS GENERALISATIONS

Arjun K. Rathie, Gradimir V. Milovanović and Richard B. Paris

Abstract. The aim of this note is to provide a natural extension and generalisation of the well-known Gelfond constant e^{π} using a hypergeometric function approach. An extension is also found for the square root of this constant. Several known mathematical constants are also deduced in hypergeometric form from our newly introduced constant.

1. Introduction and preliminaries

In mathematics, Gelfond's constant, which is named after Aleksandr Osipovich Gelfond (1906–1968), is given by e^{π}. Like both e and π, this constant is a transcendental number. The decimal expansion of Gelfond's constant is

$$\mathrm{e}^{\pi} = 23.1406926\,32779\ldots$$

and its continued fraction representation is given in [5, A039661].

This number has a connection to the Ramanujan constant $\mathrm{e}^{\pi\sqrt{163}} = (\mathrm{e}^{\pi})^{\sqrt{163}}$. It is worth noting that this last number is almost an integer:

$$\mathrm{e}^{\pi\sqrt{163}} \simeq 640320^3 + 744.$$

A geometrical occurrence of Gelfond's constant arises in the sum of even-dimension unit spheres with volume $V_{2n} = \pi^n/n!$. Then

$$\sum_{n=0}^{\infty} V_{2n} = \mathrm{e}^{\pi}.$$

There are several ways of expressing Gelfond's constant, some of which are enumerated below. We have $\mathrm{e}^{\pi} = (\mathrm{i}^{\mathrm{i}})^{-2}$, where $\mathrm{i} = \sqrt{-1}$;

$$\mathrm{e}^{\pi} = {}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix} \,\middle|\, \frac{\pi^2}{4}\right) + \pi\,{}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix} \,\middle|\, \frac{\pi^2}{4}\right),$$

2020 Mathematics Subject Classification: 11Y60, 33B10, 33C05, 33C20.
Keywords and phrases: Gelfond's constant; hypergeometric function; Gauss summation theorem.

where ${}_0F_1$ is a generalised hypergeometric function that can be expressed in terms of the modified Bessel function of the first kind of order $\mp 1/2$ (i.e., as $\pi I_{-1/2}(\pi)/\sqrt{2}$ and $I_{1/2}(\pi)/\sqrt{2}$, resp.) and finally

$$e^{\pi} = {}_2F_1\left(\begin{matrix} i, -i \\ \frac{1}{2} \end{matrix} \middle| 1\right) + 2\,{}_2F_1\left(\begin{matrix} \frac{1}{2}+i, \frac{1}{2}-i \\ \frac{3}{2} \end{matrix} \middle| 1\right), \tag{1}$$

where ${}_2F_1$ is the well-known Gauss hypergeometric function [3, p. 384]. The result (1) can be easily verified by making use of the classical Gauss summation theorem

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol and $\mathrm{Re}(c-a-b) > 0$, to evaluate the ${}_2F_1$ series on the right-hand side, combined with the identities

$$\cos \pi z = \frac{\pi}{\Gamma(\frac{1}{2}-z)\Gamma(\frac{1}{2}+z)}, \qquad \sin \pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

and Euler's formula $e^{iz} = \cos z + i \sin z$.

The extension of the summation theorem (2) to the ${}_3F_2$ hypergeometric series is available in the literature [4], which we write in the following manner:

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a, b, d+1 \\ c+1, d \end{matrix} \middle| 1\right) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_b (d+1)_n}{(c+1)_n (d)_n} \frac{1}{n!} \\ &= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)} \left\{ c-a-b+\frac{ab}{d} \right\} \end{aligned} \tag{3}$$

provided $d \neq 0, -1, -2, \ldots$ and $\mathrm{Re}(c-a-b) > 0$.

The aim of this note is to provide an extension of Gelfond's constant (1), and also its square root, with the help of the result (3). A few interesting results closely related to Gelfond's constant and its square root are also given.

2. Extension of Gelfond's constant

The natural extension of Gelfond's constant to be established here is given in the following theorem.

THEOREM 2.1. *For $d_1, d_2 \neq 0, -1, -2, \ldots$, the following result holds true:*

$$\begin{aligned} &e^{\pi}\left(\frac{1}{5d_1} + \frac{15}{32d_2} + \frac{23}{80}\right) + e^{-\pi}\left(\frac{1}{5d_1} - \frac{15}{32d_2} - \frac{7}{80}\right) \\ &\quad = {}_3F_2\left(\begin{matrix} i,\ -i,\ d_1+1 \\ \frac{3}{2},\ d_1 \end{matrix} \middle| 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+i,\ \frac{1}{2}-i,\ d_2+1 \\ \frac{5}{2},\ d_2 \end{matrix} \middle| 1\right). \end{aligned} \tag{4}$$

Proof. The derivation of (4) follows from application of the summation formula (3). We have

$${}_3F_2\left(\begin{matrix} i,\ -i,\ d_1+1 \\ \frac{3}{2},\ d_1 \end{matrix} \middle| 1\right) = (e^{\pi} + e^{-\pi})\left(\frac{1}{10} + \frac{1}{5d_1}\right)$$

and

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ d_2+1 \\ \frac{5}{2},\ d_2 \end{matrix}\ \middle|\ 1\right) = (\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})\left(\frac{3}{32}+\frac{15}{64d_2}\right). $$

Insertion of these summations into the right-hand side of (4) then yields the result asserted by the theorem. □

Corollary 2.2. *In (4), if we take $d_1 = 2/(5n-1)$ and $d_2 = 15/(2(8n-3))$ for positive integer n, then we obtain after a little calculation the following result:*

$$ n\mathrm{e}^{\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{5n+1}{5n-1} \\ \frac{3}{2},\ \frac{2}{5n-1} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{16n+9}{2(8n-3)} \\ \frac{5}{2},\ \frac{15}{2(8n-3)} \end{matrix}\ \middle|\ 1\right). \tag{5} $$

For $n = 1, 2, 3$ we find respectively the following results related to (1):

$$ \mathrm{e}^{\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{3}{2} \\ \frac{3}{2},\ \frac{1}{2} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{5}{2} \\ \frac{5}{2},\ \frac{3}{2} \end{matrix}\ \middle|\ 1\right), $$

$$ 2\mathrm{e}^{\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{11}{9} \\ \frac{3}{2},\ \frac{2}{9} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{41}{26} \\ \frac{5}{2},\ \frac{15}{26} \end{matrix}\ \middle|\ 1\right), $$

$$ 3\mathrm{e}^{\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{8}{7} \\ \frac{3}{2},\ \frac{1}{7} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{19}{14} \\ \frac{5}{2},\ \frac{5}{14} \end{matrix}\ \middle|\ 1\right). $$

We note that the expression for e^{π} reduces to (1) since both ${}_3F_2$ series contract to yield ${}_2F_1$ series.

Corollary 2.3. *In (4), if we take $d_1 = 2/(5n-1)$ and $d_2 = -15/(2(8n+3))$ for positive integer n, then we obtain after a little calculation the following result:*

$$ n\mathrm{e}^{-\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{5n+1}{5n-1} \\ \frac{3}{2},\ \frac{2}{5n-1} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{16n-9}{2(8n+3)} \\ \frac{3}{2},\ -\frac{15}{2(8n+3)} \end{matrix}\ \middle|\ 1\right). $$

In particular, for $n = 1, 2, 3$ we find respectively the following results:

$$ \mathrm{e}^{-\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{3}{2} \\ \frac{3}{2},\ \frac{1}{2} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{7}{22} \\ \frac{3}{2},\ -\frac{15}{22} \end{matrix}\ \middle|\ 1\right), $$

$$ 2\mathrm{e}^{-\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{11}{9} \\ \frac{3}{2},\ \frac{2}{9} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{23}{38} \\ \frac{3}{2},\ -\frac{15}{38} \end{matrix}\ \middle|\ 1\right), $$

$$ 3\mathrm{e}^{-\pi} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{8}{7} \\ \frac{3}{2},\ \frac{1}{7} \end{matrix}\ \middle|\ 1\right) + 2\,{}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{13}{18} \\ \frac{3}{2},\ -\frac{5}{18} \end{matrix}\ \middle|\ 1\right), $$

where the first ${}_3F_2$ series in the expression for $\mathrm{e}^{-\pi}$ contracts to yield a simpler ${}_2F_1$ series.

Similarly, if we take $d_1 = 2/(10n-1)$ and $d_2 = -5/2$ in (4), where n is a positive integer, and note that the second ${}_3F_2$ series vanishes with this choice of d_2 on account of (3), we obtain after a little calculation the following:

Corollary 2.4. *We have the result for positive integer n*

$$ n(\mathrm{e}^{\pi}+\mathrm{e}^{-\pi}) = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{10n+1}{10n-1} \\ \frac{3}{2},\ \frac{2}{10n-1} \end{matrix}\ \middle|\ 1\right). $$

In particular, for $n=1,2$ we find the following results:

$$\mathrm{e}^{\pi}+\mathrm{e}^{-\pi}={}_3F_2\left(\begin{matrix}\mathrm{i},\ -\mathrm{i},\ \frac{11}{9}\\ \frac{3}{2},\ \frac{2}{9}\end{matrix}\ \middle|\ 1\right),\quad 2(\mathrm{e}^{\pi}+\mathrm{e}^{-\pi})={}_3F_2\left(\begin{matrix}\mathrm{i},\ -\mathrm{i},\ \frac{21}{19}\\ \frac{3}{2},\ \frac{2}{19}\end{matrix}\ \middle|\ 1\right).$$

Similarly other results can be obtained.

3. The square root of Gelfond's constant: $\mathrm{e}^{\pi/2}$

Expressions for the square root of Gelfond's constant are $\mathrm{e}^{\pi/2}=\mathrm{i}^{-\mathrm{i}}$,

$$\mathrm{e}^{\pi/2}={}_2F_1\left(\begin{matrix}\mathrm{i},\ -\mathrm{i}\\ \frac{1}{2}\end{matrix}\ \middle|\ \frac{1}{2}\right)+\sqrt{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{1}{2}\right) \tag{6}$$

together with the inverse expression

$$\mathrm{e}^{-\pi/2}={}_2F_1\left(\begin{matrix}\mathrm{i},\ -\mathrm{i}\\ \frac{1}{2}\end{matrix}\ \middle|\ \frac{1}{2}\right)-\sqrt{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{1}{2}\right). \tag{7}$$

It is not out of place to mention here that the results in (6) and (7) can be obtained by evaluating the first hypergeometric function by the second Gauss theorem and the second hypergeometric function by Bailey's theorem (cf. [2])

$${}_2F_1\left(\begin{matrix}a,\ b\\ \frac{1}{2}(a+b+1)\end{matrix}\ \middle|\ \frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)},$$

$${}_2F_1\left(\begin{matrix}a,\ 1-a\\ c\end{matrix}\ \middle|\ \frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}.$$

We now derive the analogue of Theorem 2.1 by making use of the extension of the second Gauss and Bailey's theorems applied to ${}_3F_2$ series. These are given by [1]:

$$\begin{aligned}{}_3F_2\left(\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+3),\ d\end{matrix}\ \middle|\ \frac{1}{2}\right)&=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2}\right)}\\ &\times\left\{\frac{\frac{1}{2}(a+b-1)-ab/d}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}+\frac{(a+b+1)/d-2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right\},\end{aligned} \tag{8}$$

$$\begin{aligned}{}_3F_2\left(\begin{matrix}a,\ 1-a,\ d+1\\ c+1,\ d\end{matrix}\ \middle|\ \frac{1}{2}\right)&=2^{-c}\Gamma\left(\frac{1}{2}\right)\Gamma(c+1)\\ &\times\left\{\frac{2/d}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}+\frac{1-(c/d)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+1\right)}\right\},\end{aligned} \tag{9}$$

provided $d\neq 0,-1,-2,\ldots$. Then we have the following theorem.

THEOREM 3.1. *For $d_1,d_2\neq 0,-1,-2,\ldots$, the following result holds true:*

$$\begin{aligned}&\mathrm{e}^{\pi/2}\left(\frac{1}{10d_1}+\frac{3}{16d_2}+\frac{27}{40}\right)+\mathrm{e}^{-\pi/2}\left(\frac{3}{10d_1}-\frac{21}{16d_2}+\frac{11}{40}\right)\\ &={}_3F_2\left(\begin{matrix}\mathrm{i},\ -\mathrm{i},\ d_1+1\\ \frac{3}{2},\ d_1\end{matrix}\ \middle|\ \frac{1}{2}\right)+\sqrt{2}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ d_2+1\\ \frac{5}{2},\ d_2\end{matrix}\ \middle|\ \frac{1}{2}\right).\end{aligned} \tag{10}$$

Proof. In the first ${}_3F_2$ series use (8) and in the second ${}_3F_2$ series use (9) together with standard properties of the gamma function. ☐

COROLLARY 3.2. *If in* (10) *we take* $d_1 = 1/(7n-5)$ *and* $d_2 = 15/(24n-14)$ *for positive integer* n *then we find*

$$ne^{\pi/2} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{7n-4}{7n-5} \\ \frac{3}{2},\ \frac{1}{7n-5} \end{matrix} \,\middle|\, \frac{1}{2}\right) + \sqrt{2}\ {}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{24n+1}{24n-14} \\ \frac{5}{2},\ \frac{15}{24n-14} \end{matrix} \,\middle|\, \frac{1}{2}\right).$$

When $n = 1$ we recover (6) after contraction of the ${}_3F_2$ series. For $n = 2, 3$ we find respectively the following results:

$$2e^{\pi/2} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{10}{9} \\ \frac{3}{2},\ \frac{1}{9} \end{matrix} \,\middle|\, \frac{1}{2}\right) + \sqrt{2}\ {}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{49}{34} \\ \frac{5}{2},\ \frac{15}{34} \end{matrix} \,\middle|\, \frac{1}{2}\right),$$

$$3e^{\pi/2} = {}_3F_2\left(\begin{matrix} \mathrm{i},\ -\mathrm{i},\ \frac{17}{16} \\ \frac{3}{2},\ \frac{1}{16} \end{matrix} \,\middle|\, \frac{1}{2}\right) + \sqrt{2}\ {}_3F_2\left(\begin{matrix} \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i},\ \frac{73}{58} \\ \frac{5}{2},\ \frac{15}{58} \end{matrix} \,\middle|\, \frac{1}{2}\right).$$

Similarly other results can be obtained.

4. Generalisations of Gelfond's constant

The natural generalisation of Gelfond's constant established in this section is given by the following theorem.

THEOREM 4.1. *For* $\mu > 0$ *and* ν *an arbitrary complex parameter with* $\lambda = \pi^{-1}\ln\mu + \nu$, *the following result holds true:*

$$\mu e^{\pi\nu} = e^{\pi\lambda} = {}_2F_1\left(\begin{matrix} \mathrm{i}\lambda,\ -\mathrm{i}\lambda \\ \frac{1}{2} \end{matrix} \,\middle|\, 1\right) + 2\lambda\,{}_2F_1\left(\begin{matrix} \frac{1}{2}+\mathrm{i}\lambda,\ \frac{1}{2}-\mathrm{i}\lambda \\ \frac{3}{2} \end{matrix} \,\middle|\, 1\right) \tag{11}$$

Proof. The proof of (11) follows along similar lines to that outlined for (1). ☐

REMARK 4.2. Other ways of expressing $e^{\pi\lambda}$ are:

$$e^{\pi\lambda} = (\mathrm{i}^{\mathrm{i}})^{-2\lambda} \qquad (\mathrm{i} = \sqrt{-1}); \quad e^{\pi\lambda} = {}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix} \,\middle|\, \frac{\pi^2\lambda^2}{4}\right) + \pi\lambda\ {}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix} \,\middle|\, \frac{\pi^2\lambda^2}{4}\right),$$

REMARK 4.3. The result (11) yields the alternative expression for $e^{\pm\pi/2}$ given by

$$e^{\pm\pi/2} = {}_2F_1\left(\begin{matrix} \frac{1}{2}\mathrm{i},\ -\frac{1}{2}\mathrm{i} \\ \frac{1}{2} \end{matrix} \,\middle|\, 1\right) \pm {}_2F_1\left(\begin{matrix} \frac{1}{2}+\frac{1}{2}\mathrm{i},\ \frac{1}{2}-\frac{1}{2}\mathrm{i} \\ \frac{3}{2} \end{matrix} \,\middle|\, 1\right).$$

Here we mention several mathematical constants that are derivable from our new constant $\mu e^{\pi\nu}$ by suitable choice of the parameters μ and ν. Each constant can consequently be expressed in terms of Gauss hypergeometric functions by (11). These constants are enumerated below.

If we choose $\mu = 1$, $\nu = \sqrt{19}$, $\sqrt{43}$, $\sqrt{67}$ and $\sqrt{163}$ in (11), we obtain hypergeometric function representations of the four largest Heegner numbers, viz.

$$e^{\pi\sqrt{19}} = 8.85479\,77768\,01543\,19497\ldots \times 10^5 \ \doteq 96^3 + 744 - 0.22;$$

$$\mathrm{e}^{\pi\sqrt{43}} = 8.84736\,74399\,97774\,66034\ldots \times 10^{8} \doteq 960^3 + 744 - 0.00022;$$
$$\mathrm{e}^{\pi\sqrt{67}} = 1.47197\,95274\,39999\,98662\ldots \times 10^{11} \doteq 5280^3 + 744 - 0.0000013;$$
$$\mathrm{e}^{\pi\sqrt{163}} = 2.62537\,41264\,07687\,43999\ldots \times 10^{17} \doteq 640320^3 + 744 - 7.5 \times 10^{-13},$$

the last number being Ramanujan's constant.

Other choices for μ and ν in (11) yield hypergeometric representations of the following constants:

$\mu = 1,\ \nu = 1:$ Gelfond $\mathrm{e}^{\pi} = 23.14069\ 26327\ 79269\ 00572\ldots;$

$\mu = \pi^2/8,\ \nu = 0:$ Favard $\frac{3}{4}\zeta(2) = 1.23370\ 05501\ 36169\ 82735\ldots;$

$\mu = \pi^2/12,\ \nu = 0:$ Nielsen-Ramanujan $\frac{1}{2}\zeta(2) = 0.82246\ 70334\ 24113\ 21823\ldots;$

$\mu = \dfrac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)},\ \nu = 0:$ Bloch-Landau $L = 0.54325\ 89653\ 42976\ 70695\ldots;$

$\mu = 2^{\sqrt{2}},\ \nu = 0:$ Gelfond-Schneider $G_{\mathrm{GS}} = 2.66514\ 41426\ 90225\ 18865\ldots;$

$\mu = \dfrac{\pi^2}{12\log 2},\ \nu = 0:$ Khinchin-Lévy $\beta = 1.18656\ 91104\ 15625\ 45282\ldots;$

$\mu = \dfrac{4\sqrt{\pi}}{2^{3/4}\Gamma^2\left(\frac{1}{4}\right)},\ \nu = \dfrac{1}{8}:$ Weierstrass $\sigma\left(\frac{1}{2}\right) = 0.47494\ 93799\ 87920\ 65033\ldots;$

$\mu = \dfrac{\pi}{\log 2},\ \nu = 0:$ Van der Pauw $\alpha = 4.53236\ 01418\ 27193\ 80962\ldots;$

$\mu = \dfrac{6}{\pi^2}\log 2\,\log 10,\ \nu = 0:$ Loche $\mathcal{L}_{\mathrm{Lo}} = 0.97027\ 01143\ 92033\ 92574\ldots;$

$\mu = \dfrac{1}{4\pi^{3/2}}\Gamma^2\left(\dfrac{1}{4}\right),\ \nu = 0:$ Chebyshev $\lambda_{\mathrm{Ch}} = 0.59017\ 02995\ 08048\ 11302\ldots.$

Similarly several other constants available in the literature can be easily found from (11).

Acknowledgement. The work of G.V. Milovanović was supported in part by the Serbian Academy of Sciences and Arts (Project Φ-96).

References

[1] Y.S. Kim, M.A. Rakha, A.K. Rathie, *Extensions of certain classical summation theorems for the series* $_2F_1$, $_3F_2$ *and* $_4F_3$ *with applications in Ramanujan's summations*, Int. J. Math. Sci. **2010**, Article ID 309503, 26 pages.

[2] G.V. Milovanović, R.K. Parmar, A.K. Rathie, *A study of generalized summation theorems for the series* $_2F_1$ *with an applications to Laplace transforms of convolution type integrals involving Kummer's functions* $_1F_1$, Appl. Anal. Discrete Math. **12** (2018), 257–272.

[3] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[4] A.P. Prudnikov, Yu. A. Brychkov, O.I. Marichev, *Integrals and Series: Special Functions*, Vol. 3, Gordon and Breach, New York, 1988.

[5] N.J.A. Sloane, *On-line Encyclopedia of Integer Sequences*, 2008 (`https://oeis.org`).

(received 10.03.2021; in revised form 29.04.2021; available online 31.12.2021)

Department of Mathematics, Vedant College of Engineering and Technology, Rajasthan Technical University, Bundi, 323021, Rajasthan, India

E-mail: arjunkumarrathie@gmail.com

Serbian Academy of Sciences and Arts, 11000 Beograd, Serbia

Faculty of Science and Mathematics, University of Niš, 18000 Niš, Serbia

E-mail: gvm@mi.sanu.ac.rs

Division of Computing and Mathematics, Abertay University, Dundee DD1 1HG, UK

E-mail: r.paris@abertay.ac.uk