Certain transformations and summations for generalized hypergeometric series with integral parameter differences

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Abstract

Certain transformation and summation formulas for generalized hypergeometric series with integral parameter differences are derived.

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1. Introduction

The generalized hypergeometric function ${}_pF_q(x)$ is defined for complex parameters and argument by the series

$${}_pF_q\left(\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}\,\middle|\, x\right) = \sum_{k=0}^\infty \frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k(b_2)_k\ldots(b_q)_k}\,\frac{x^k}{k!}. \tag{1.1}$$

When $q = p$ this series converges for $|x| < \infty$, but when $q = p-1$ convergence occurs when $|x| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. In (1.1) the Pochhammer symbol, or ascending factorial, $(a)_k$ is defined by $(a)_0 = 1$ and for $k \geq 1$ by $(a)_k = a(a+1)\ldots(a+k-1)$. However, for all integers k we write simply

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

In what follows we shall adopt the convention of writing the finite (except where noted otherwise) sequence of parameters $(a_1, \ldots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k \equiv (a_1)_k \ldots (a_p)_k,$$

with an empty product $p = 0$ reducing to unity.

In [1–4] we derived in various ways transformation formulas for the generalized hypergeometric functions ${}_{r+1}F_{r+1}(x)$ and ${}_{r+2}F_{r+1}(x)$, where here and below at least r pairs of numeratorial and denominatorial parameters differ by arbitrary positive integers. In particular, in [4] we stated without proof that the generalized hypergeometric function ${}_{r+1}F_{r+1}(x)$ in which $r+1$ pairs of numeratorial and denominatorial parameters differ by arbitrary positive integers may be written as a product of e^x and a certain polynomial in x. In Section 3 we shall provide a proof of this result and discuss its implications.

In [1, 4, 5] we showed that essentially the same methods used to obtain the transformation formulas alluded to above may be employed to deduce the Karlsson-Minton and other more general summation formulas for the ${}_{r+2}F_{r+1}(1)$ generalized hypergeometric series with unit argument. In the present investigation we shall derive in Section 4 in a similar manner analogous summation formulas for generalized hypergeometric series of the type ${}_{r+2}F_{r+1}(\frac{1}{2})$ with half unit argument. Finally, in Section 5 we shall consider a certain quadratic transformation for the ${}_3F_2(x)$ hypergeometric function.

2. Preliminary results

We record two lemmas and a theorem that we shall utilize in the sequel. Lemmas 1 and 2 are proved respectively in [1] and [4] and Theorem 1 is proved in [4]. The notation $\{{n \atop k}\}$ will be employed to denote the Stirling numbers of the second kind. These nonnegative integers represent the number of ways to partition n objects into k nonempty sets and arise for nonnegative integers n in the generating relation [6, p. 262]

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^k (-x)_k, \qquad \left\{ {n \atop 0} \right\} = \delta_{0n},$$

where δ_{0n} is the Kronecker symbol.

Lemma 1. *For nonnegative integers j define*

$$S_j \equiv \sum_{n=0}^{\infty} n^j \frac{\lambda_n}{n!}, \qquad S_0 \equiv \sum_{n=0}^{\infty} \frac{\lambda_n}{n!},$$

where the infinite sequence (λ_n) is such that S_j converges for all j. Then

$$S_j = \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \sum_{n=0}^{\infty} \frac{\lambda_{n+k}}{n!}.$$

Lemma 2. *For nonnegative integer s let (a_s) denote a parameter sequence containing s elements, where when $s=0$ the sequence is empty. Let (a_s+k) denote the sequence when k is added to each element of (a_s). Let $\mathcal{F}(x)$ denote the generalized hypergeometric function with r numeratorial and denominatorial parameters differing by the positive integers (m_r), namely*

$$\mathcal{F}(x) \equiv {}_{r+s}F_{r+1} \left(\begin{matrix} (a_s), & (f_r+m_r) \\ c, & (f_r) \end{matrix} \middle| \, x \right),$$

where by (1.1) convergence of the series representation for the latter occurs in an appropriate domain depending on the values of s and the elements of the parameter sequence (a_s). Then

$$\mathcal{F}(x) = \frac{1}{A_0} \sum_{k=0}^{m} x^k A_k \frac{((a_s))_k}{(c)_k} \, {}_sF_1 \left(\begin{matrix} (a_s+k) \\ c+k \end{matrix} \middle| \, x \right),$$

where $m = m_1 + \cdots + m_r$, the coefficients A_k are defined by

$$A_k \equiv \sum_{j=k}^{m} \left\{ {j \atop k} \right\} \sigma_{m-j}, \qquad A_0 = (f_1)_{m_1} \ldots (f_r)_{m_r}, \quad A_m = 1 \tag{2.1}$$

and the σ_j $(0 \le j \le m)$ *are generated by the relation*

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} x^j. \tag{2.2}$$

Theorem 1. *Let* (m_r) *be a nonempty sequence of positive integers and define* $m \equiv m_1 + \cdots + m_r$*. Then if* $b \neq f_j$ $(1 \le j \le r)$*,* $(\lambda)_m \neq 0$*, where* $\lambda \equiv c - b - m$*, we have the transformation formula*

$${}_{r+1}F_{r+1}\left(\begin{array}{c} b,\ (f_r + m_r) \\ c,\quad (f_r) \end{array}\middle|\, x\right) = e^x\ {}_{m+1}F_{m+1}\left(\begin{array}{c} \lambda,\ (\xi_m + 1) \\ c,\quad (\xi_m) \end{array}\middle|\, -x\right), \tag{2.3}$$

where $|x| < \infty$*. The* (ξ_m) *are the nonvanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* m *given by*

$$Q_m(t) = \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} \left\{ \begin{array}{c} j \\ k \end{array} \right\} (b)_k (t)_k (\lambda - t)_{m-k},$$

where the σ_j $(0 \le j \le m)$ *are determined by the generating relation (2.2).*

In the following Section 3 we shall consider the generalized hypergeometric function $w(x)$ defined for $|x| < \infty$ by

$$w(x) \equiv {}_{r+1}F_{r+1}\left(\begin{array}{c} (f_{r+1} + m_{r+1}) \\ (f_{r+1}) \end{array}\middle|\, x\right),$$

where (m_{r+1}) is a sequence of positive integers. It is evident that $w(x)$ is an entire function.

3. Properties of $w(x)$

If in Theorem 1 we set $b = f_{r+1} + m_{r+1}$, $c = f_{r+1}$ and define $m \equiv m_1 + \cdots + m_r$, $M \equiv m + m_{r+1}$, then $\lambda = -M$, $(-M)_m \neq 0$ and we find from (2.3) that

$$w(x) = e^x\ {}_{m+1}F_{m+1}\left(\begin{array}{c} -M,\ (\xi_m + 1) \\ f_{r+1},\quad (\xi_m) \end{array}\middle|\, -x\right).$$

The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial of degree m given by

$$Q_m(t) = \sum_{j=0}^{m} \sigma_{m-j} \sum_{k=0}^{j} \left\{ \begin{array}{c} j \\ k \end{array} \right\} (f_{r+1} + m_{r+1})_k (t)_k (-M - t)_{m-k},$$

where the σ_j $(0 \le j \le m)$ are generated by (2.2).

Thus it is evident that $w(x)$ is proportional to a polynomial in x of degree at most M which we define as

$$\mathcal{P}_M(x) \equiv {}_{m+1}F_{m+1}\left(\begin{array}{c} -M,\ (\xi_m + 1) \\ f_{r+1},\quad (\xi_m) \end{array}\middle|\, -x\right).$$

Moreover, since e^x can never vanish it follows that the entire function $w(x)$ has at most M zeros in the complex plane. However, we shall obtain an explicit representation for $\mathcal{P}_M(x)$ which shows that its degree is exactly M.

Theorem 2. *Let* (m_{r+1}) *be a sequence of positive integers such that* $M \equiv m_1 + \cdots + m_{r+1}$ *and let* (f_{r+1}) *be a sequence of complex numbers such that* $(f_1)_{m_1} \ldots (f_{r+1})_{m_{r+1}}$ *is nonvanishing. Then*

$$w(x) = e^x \mathcal{P}_M(x),$$

where $\mathcal{P}_M(x)$ is a polynomial of degree M given by

$$\mathcal{P}_M(x) = \frac{1}{B_0}\sum_{k=0}^{M} B_k x^k, \qquad B_k \equiv \sum_{j=k}^{M} \left\{ {j \atop k} \right\} \rho_{M\text{-}j}\,.$$

Here

$$B_0 = (f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}, \qquad B_M = 1,$$

and the ρ_j $(0 \le j \le M)$ are generated by the relation

$$(f_1 + x)_{m_1} \dots (f_{r+1} + x)_{m_{r+1}} = \sum_{j=0}^{M} \rho_{M\text{-}j} x^j.$$

Proof: Note that

$$\frac{((f_{r+1} + m_{r+1}))_n}{((f_{r+1}))_n} = \frac{(f_1 + n)_{m_1}}{(f_1)_{m_1}} \cdots \frac{(f_{r+1} + n)_{m_{r+1}}}{(f_r)_{m_{r+1}}},$$

where the numeratorial expression on the right-hand side of the latter may be written as

$$(f_1 + n)_{m_1} \dots (f_{r+1} + n)_{m_{r+1}} = \sum_{j=0}^{M} \rho_{M\text{-}j}\, n^j. \tag{3.1}$$

Thus by (1.1)

$$w(x) \equiv {}_{r+1}F_{r+1}\left(\begin{matrix} (f_{r+1} + m_{r+1}) \\ (f_{r+1}) \end{matrix} \,\middle|\, x \right) = \sum_{n=0}^{\infty} \frac{((f_{r+1} + m_{r+1}))_n}{((f_{r+1}))_n} \frac{x^n}{n!}$$

$$= \frac{1}{(f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}} \sum_{j=0}^{M} \rho_{M\text{-}j} \sum_{n=0}^{\infty} n^j \frac{x^n}{n!},$$

where the order of the summations has been interchanged.

Now employing Lemma 1 we have

$$\sum_{n=0}^{\infty} n^j \frac{x^n}{n!} = \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \sum_{n=0}^{\infty} \frac{x^{n+k}}{n!} = e^x \sum_{k=0}^{j} \left\{ {j \atop k} \right\} x^k$$

so that

$$w(x) = \frac{e^x}{(f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}} \sum_{j=0}^{M} \rho_{M\text{-}j} \sum_{k=0}^{j} \left\{ {j \atop k} \right\} x^k,$$

where

$$\sum_{j=0}^{M} \rho_{M\text{-}j} \sum_{k=0}^{j} \left\{ {j \atop k} \right\} x^k = \sum_{k=0}^{M} \left(\sum_{j=k}^{M} \left\{ {j \atop k} \right\} \rho_{M\text{-}j} \right) x^k.$$

Defining

$$B_k \equiv \sum_{j=k}^{M} \left\{ {j \atop k} \right\} \rho_{M\text{-}j} \qquad (0 \le k \le M),$$

we note that when $k = 0$ the only contribution to the j-summation comes from $j = 0$, so that by using (3.1) we find

$$B_0 = \rho_M = (f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}.$$

In addition when $k = M$, the only contribution to the latter summation comes from $j = M$, so that by again using (3.1) we see that

$$B_M = \rho_0 = 1.$$

Thus

$$w(x) = \frac{e^x}{B_0} \sum_{k=0}^{M} B_k x^k \tag{3.2}$$

which evidently completes the proof. □

Thus we also have the following.

Corollary 1. *The entire function $w(x)$ has exactly M zeros in the complex plane.*

We remark that Ki and Kim [7] only show the existence of at most M zeros for $w(x)$, whereas from (3.2) we can in principle obtain all of the M zeros. For example,

$${}_2F_2\left(\begin{matrix} f+1,\ g+1 \\ f,\quad g \end{matrix}\,\middle|\, x\right) = e^x \left(\frac{1}{fg}x^2 + \frac{f+g+1}{fg}x + 1\right),$$

and here the zeros of $\mathcal{P}_2(x)$ are

$$x_{1,2} = -\tfrac{1}{2}(f+g+1 \pm [(f-g)^2 + 2f + 2g + 1]^{1/2}).$$

4. Summation formulas for ${}_{r+2}F_{r+1}(\frac{1}{2})$

In [4] we employed Lemma 2 to readily obtain a generalization of the Karlsson-Minton summation formula which is given in the following.

Theorem 3. *Suppose (m_r) is a sequence of positive integers such that $m = m_1 + \cdots + m_r$. Then provided that $Re\,(c-a-b) > m$ we have*

$${}_{r+2}F_{r+1}\left(\begin{matrix} a, b,\ (f_r+m_r) \\ c,\quad (f_r) \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \frac{A_k}{A_0} \frac{(-1)^k (a)_k (b)_k}{(1+a+b-c)_k},$$

where the A_k $(0 \le k \le m)$ are given by (2.1).

However we may also utilize Lemma 2 to obtain summation formulas for ${}_{r+2}F_{r+1}(\frac{1}{2})$. To this end we note the following (see, for example, [8, Section 7.3.7 (3)–(6)]) summations for ${}_2F_1(\frac{1}{2})$ given by

$${}_2F_1\left(\begin{matrix} a,\ b \\ \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2} \end{matrix}\,\middle|\, \frac{1}{2}\right) = \sqrt{\pi}\left\{\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})} + \frac{2\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2})}{\Gamma(a)\Gamma(b)}\right\}, \tag{4.1}$$

$${}_2F_1\left(\begin{matrix} a,\ b \\ \frac{1}{2}a+\frac{1}{2}b \end{matrix}\,\middle|\, \frac{1}{2}\right) = \sqrt{\pi}\left\{\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})} + \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}\right\}, \tag{4.2}$$

$${}_2F_1\left(\begin{matrix} a,\ b \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2} \end{matrix}\,\middle|\, \frac{1}{2}\right) = \sqrt{\pi}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}, \tag{4.3}$$

$${}_2F_1\left(\begin{matrix} a,\ b \\ \frac{1}{2}a+\frac{1}{2}b+1 \end{matrix}\,\middle|\, \frac{1}{2}\right) = \frac{2\sqrt{\pi}}{a-b}\left\{\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})}\right\}. \tag{4.4}$$

Setting $x = \frac{1}{2}$, $s = 2$, $(a_s) = (a, b)$, $c = \frac{1}{2}(a + b + n)$, where $n = -1, 0, 1, 2$, in the first two equations of Lemma 2, we see that

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a,\, b, & (f_r+m_r) \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}n, & (f_r) \end{matrix} \,\middle|\, \frac{1}{2} \right)
= \sum_{k=0}^{m} \left(\frac{1}{2}\right)^k \frac{A_k}{A_0} \frac{(a)_k (b)_k}{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}n)_k} \, {}_2F_1\left(\begin{matrix} a+k, b+k \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}n+k \end{matrix} \,\middle|\, \frac{1}{2} \right), \tag{4.5}
$$

where when $r = 0$, (f_r) and $(f_r + m_r)$ are empty and we define $m \equiv 0$. Thus the Gauss series ${}_2F_1(\frac{1}{2})$ in (4.5) with respectively $n = -1, 0, 1, 2$ correspond to the left-hand sides of (4.1)–(4.4) with $a \mapsto a + k$ and $b \mapsto b + k$. Combining these with (4.5) we therefore deduce the following.

Theorem 4. *Let (m_r) be an arbitrary sequence of positive integers such that $m \equiv m_1 + \cdots + m_r$ and (f_r) a sequence of complex numbers such that $(f_1)_{m_1} \ldots (f_r)_{m_r} \neq 0$. Then*

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a,\, b, & (f_r+m_r) \\ \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}, & (f_r) \end{matrix} \,\middle|\, \frac{1}{2} \right)
= \frac{2\sqrt{\pi}}{A_0} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} \left(\frac{1}{2}\right)^k A_k \left\{ 1 + \frac{\Gamma(a+k)\Gamma(b+k)(a+b-1+2k)}{4\Gamma(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}k)\Gamma(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}k)} \right\}, \tag{4.6}
$$

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a,\, b, & (f_r+m_r) \\ \frac{1}{2}a+\frac{1}{2}b, & (f_r) \end{matrix} \,\middle|\, \frac{1}{2} \right)
= \frac{2^{a+b-2}}{A_0\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} 2^k A_k \left\{ \Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}+\tfrac{1}{2}k) - \Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}+\tfrac{1}{2}k) \right\}, \tag{4.7}
$$

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a,\, b, & (f_r+m_r) \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}, & (f_r) \end{matrix} \,\middle|\, \frac{1}{2} \right)
= \frac{2^{a+b-2}}{A_0\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} 2^k A_k \, \Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k), \tag{4.8}
$$

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a,\, b, & (f_r+m_r) \\ \frac{1}{2}a+\frac{1}{2}b+1, & (f_r) \end{matrix} \,\middle|\, \frac{1}{2} \right)
= \frac{2^{a+b-2}}{A_0\sqrt{\pi}} \frac{a+b}{a-b} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} 2^k A_k \left\{ \Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k+\tfrac{1}{2}) - \Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k+\tfrac{1}{2}) \right\}, \tag{4.9}
$$

where the A_k $(0 \leq k \leq m)$ are given by (2.1).

We have used the duplication formula

$$
\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \tag{4.10}
$$

to obtain (4.7)–(4.9). Thus, when $r = 0$, the sequences (f_r) and $(f_r + m_r)$ are empty so that $m = 0$ and (4.6)–(4.9) reduce respectively to (4.1)–(4.4). Since (4.9) is not valid when $a = b$ we may use l'Hôpital's rule and (4.10) to obtain the limiting case of (4.9), namely

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a,\, a, & (f_r+m_r) \\ a+1, & (f_r) \end{matrix} \,\middle|\, \frac{1}{2} \right) = \frac{2^{a-1}a}{A_0} \sum_{k=0}^{m} A_k (a)_k \left\{ \psi(\tfrac{1}{2}a+\tfrac{1}{2}k+\tfrac{1}{2}) - \psi(\tfrac{1}{2}a+\tfrac{1}{2}k) \right\},
$$

where ψ is the digamma or psi function. When $r=0$ we retrieve

$$ {}_2F_1\left(\begin{matrix} a,\, a \\ a+1 \end{matrix}\,\middle|\,\tfrac{1}{2}\right) = 2^{a-1}a\left\{\psi(\tfrac{1}{2}a+\tfrac{1}{2})-\psi(\tfrac{1}{2}a)\right\} $$

which is found in [8, Section 7.3.7 (16)] in an equivalent form.

In addition, if we let $r=1$, $m_1=1$ and $f_1=f$, so that $A_0=f$, $A_1=1$, then we obtain from (4.9) and (4.10) after some straightforward algebra the result (when $a\neq b$)

$$ {}_3F_2\left(\begin{matrix} a,\, b, & f+1 \\ \frac{1}{2}a+\frac{1}{2}b+1, & f \end{matrix}\,\middle|\,\tfrac{1}{2}\right) = \sqrt{\pi}\left(\frac{a+b}{a-b}\right)\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}b)\left\{\frac{(a-f)/f}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})}-\frac{(b-f)/f}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}\right\}, $$

which has been obtained by Rathie and Pogány in [9].

5. A quadratic transformation for ${}_3F_2(x)$

In [4] by utilizing Lemma 2 we derived two quadratic transformations for

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, a+\frac{1}{2}, & (f_r+m_r) \\ c, & (f_r) \end{matrix}\,\middle|\,X\right), $$

where $X\equiv x^2/(1\pm x)^2$ and $X\equiv 4x/(1+x)^2$. As discussed below certain parametric conditions guaranteeing the existence of a quadratic transformation for the Gauss function ${}_2F_1(x)$ restrict the existence of more general quadratic transformations for ${}_{r+2}F_{r+1}(x)$ when the decomposition Lemma 2 is employed. We illustrate this by deducing a quadratic transformation that necessarily is restricted to $r=1$ thus giving the quadratic transformation for ${}_3F_2(x)$ in the following theorem.

Theorem 5. *Suppose $a,\, b\neq\frac{1}{2}$ and $f\neq a+b+\frac{1}{2}$. Then*

$$ {}_3F_2\left(\begin{matrix} a,\, b, & f+1 \\ a+b+\frac{1}{2}, & f \end{matrix}\,\middle|\,4x(1-x)\right) $$

$$ = (1-2x)^{-1}\,{}_4F_3\left(\begin{matrix} 2a-1,\, 2b-1, & \xi_1+1, & \xi_2+1 \\ a+b+\frac{1}{2}, & \xi_1, & \xi_2 \end{matrix}\,\middle|\,x\right), \qquad (5.1) $$

where ξ_1, ξ_2 are given by

$$ \xi_{1,2}=\alpha+\tfrac{1}{2}\pm[(\alpha+\tfrac{1}{2})^2-2\alpha f]^{1/2}, \qquad \alpha=\frac{(2a-1)(2b-1)}{2(a+b-f)-1}. \qquad (5.2) $$

The transformation (5.1) holds in a neighborhood of $x=0$.

Proof: From Lemma 2 we have the expansion of ${}_{r+2}F_{r+1}(z)$, with r pairs of parameters differing by the positive integers (m_r), as a finite sum of ${}_2F_1(z)$ functions in the form

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix}\,\middle|\,z\right) = \frac{1}{A_0}\sum_{k=0}^{m} z^k A_k\,\frac{(a)_k(b)_k}{(c)_k}\,{}_2F_1\left(\begin{matrix} a+k, b+k \\ c+k \end{matrix}\,\middle|\,z\right), \qquad (5.3) $$

where the coefficients A_k are defined by (2.1). If we set $r=1$, $m_1=1$ and $f_1=f$ in the above expansion we obtain the particular case

$$ {}_3F_2\left(\begin{matrix} a, b, & f+1 \\ c, & f \end{matrix}\,\middle|\,z\right) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\,z\right)+\frac{abz}{cf}\,{}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}\,\middle|\,z\right). \qquad (5.4) $$

Now let $z = X \equiv 4x(1-x)$ and define

$$F(x) \equiv {}_3F_2\left(\begin{matrix} a,\ b, & f+1 \\ a+b+\frac{1}{2}, & f \end{matrix}\,\middle|\, X\right).$$

Then, from (5.4) with $c = a+b+\frac{1}{2}$, we find

$$F(x) = {}_2F_1\left(\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}\,\middle|\, X\right) + \frac{ab}{cf}\, X\, {}_2F_1\left(\begin{matrix} a+1, b+1 \\ a+b+\frac{3}{2} \end{matrix}\,\middle|\, X\right). \tag{5.5}$$

We now make use of two quadratic transformation formulas for the Gauss hypergeometric function [10, Section 15.3, (22) and (24)], namely

$${}_2F_1\left(\begin{matrix} a, b \\ a+b-j+\frac{1}{2} \end{matrix}\,\middle|\, X\right) = (1-2x)^{-j}\, {}_2F_1\left(\begin{matrix} 2a-j, 2b-j \\ a+b-j+\frac{1}{2} \end{matrix}\,\middle|\, x\right), \tag{5.6}$$

where $j = 0, 1$. Application of these transformations with $j = 0$ and $j = 1$ respectively to the first and second hypergeometric functions in (5.5) then leads to

$$F(x) = (1-2x)^{-1}\left\{(1-2x)\, {}_2F_1\left(\begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix}\,\middle|\, x\right) + \right.$$

$$\left. \frac{4ab}{cf}\, x(1-x)\, {}_2F_1\left(\begin{matrix} 2a+1, 2b+1 \\ a+b+\frac{3}{2} \end{matrix}\,\middle|\, x\right)\right\}. \tag{5.7}$$

From (1.1) it can be seen that for arbitrary values of c

$$\begin{aligned} {}_2F_1\left(\begin{matrix} 2a, 2b \\ c \end{matrix}\,\middle|\, x\right) &= \sum_{n=0}^{\infty} \frac{(2a)_n (2b)_n}{(c)_n} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{(2a-1+n)(2b-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

$$\begin{aligned} x\, {}_2F_1\left(\begin{matrix} 2a, 2b \\ c \end{matrix}\,\middle|\, x\right) &= \sum_{n=1}^{\infty} \frac{(2a)_{n-1} (2b)_{n-1}}{(c)_{n-1}} \frac{x^n}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{n(c-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

$$\begin{aligned} x\, {}_2F_1\left(\begin{matrix} 2a+1, 2b+1 \\ c+1 \end{matrix}\,\middle|\, x\right) &= \sum_{n=1}^{\infty} \frac{(2a+1)_{n-1} (2b+1)_{n-1}}{(c+1)_{n-1}} \frac{x^n}{(n-1)!} \\ &= \frac{c}{4ab} \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{n(2a-1+n)(2b-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

and

$$\begin{aligned} x^2\, {}_2F_1\left(\begin{matrix} 2a+1, 2b+1 \\ c+1 \end{matrix}\,\middle|\, x\right) &= \sum_{n=2}^{\infty} \frac{(2a+1)_{n-2} (2b+1)_{n-2}}{(c+1)_{n-2}} \frac{x^n}{(n-2)!} \\ &= \frac{c}{4ab} \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{n(n-1)(c-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

where obvious adjustments to the summation index n have been made. Then, upon setting $c = a + b + \frac{1}{2}$ in the above sums, we obtain from (5.7)

$$F(x) = \frac{(1-2x)^{-1}}{(2a-1)(2b-1)} \sum_{n=0}^{\infty} \frac{(2a-1)_n(2b-1)_n}{(a+b+\frac{1}{2})_n} \frac{x^n}{n!} P_2(n),$$

where, with $\gamma \equiv f^{-1}(a+b-\frac{1}{2}) - 1$,

$$\begin{aligned} P_2(n) &= (2a-1+n)(2b-1+n)\left(1+\frac{n}{f}\right) - n\left(2+\frac{n-1}{f}\right)(a+b-\tfrac{1}{2}+n) \\ &= \gamma n^2 + \left(\gamma + \frac{1}{f}(2a-1)(2b-1)\right)n + (2a-1)(2b-1). \end{aligned} \tag{5.8}$$

The quadratic $P_2(n)$ can be factored to yield

$$P_2(n) = \gamma(n+\xi_1)(n+\xi_2),$$

where, provided $a, b \neq \frac{1}{2}$ and $f \neq a+b-\frac{1}{2}$, ξ_1 and ξ_2 are given by (5.2). Since $\gamma\xi_1\xi_2 = (2a-1)(2b-1)$, we therefore see that

$$\begin{aligned} F(x) &= (1-2x)^{-1} \sum_{n=0}^{\infty} \frac{(2a-1)_n(2b-1)_n}{(a+b+\frac{1}{2})_n} \frac{x^n}{n!} \frac{(n+\xi_1)(n+\xi_2)}{\xi_1\,\xi_2} \\ &= (1-2x)^{-1} \sum_{n=0}^{\infty} \frac{(2a-1)_n(2b-1)_n}{(a+b+\frac{1}{2})_n} \frac{(\xi_1+1)_n(\xi_2+1)_n}{(\xi_1)_n\,(\xi_2)_n} \frac{x^n}{n!}, \end{aligned}$$

where we have made use of the fact that $(\lambda+1)_n/(\lambda)_n = (n+\lambda)/\lambda$. This yields (5.1) and so completes the proof. □

We remark that when $f = a+b-\frac{1}{2}$, the coefficient $\gamma = 0$ and the polynomial $P_2(n)$ becomes linear with $\xi_1 = f$. The formula (5.1) then reduces to the quadratic transformation (5.6) with $j = 1$.

If we try to apply the same reasoning to the more general function with r pairs of parameters differing by the positive integers (m_r)

$${}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ a+b+\frac{1}{2}, & (f_r) \end{matrix} \,\middle|\, X\right), \tag{5.9}$$

we obtain from (5.3) a series of $m+1$ terms involving the Gauss functions

$$F_k \equiv {}_2F_1\left(\begin{matrix} a+k, b+k \\ a+b+k+\frac{1}{2} \end{matrix} \,\middle|\, X\right) \qquad (0 \leq k \leq m).$$

A quadratic transformation for ${}_2F_1(\alpha, \beta; \gamma \,|\, x)$ exists if and only if any of the quantities

$$\pm(1-\gamma), \quad \pm(\alpha-\beta), \quad \pm(\alpha+\beta-\gamma)$$

are such that either one of them equals $\frac{1}{2}$ or two of them are equal [10, p. 560]. In this case, the third condition above for the functions F_k, with $0 \leq k \leq m$, has the form $\alpha+\beta-\gamma = k-\frac{1}{2}$; that is, a quadratic transformation exists only when $k = 0$ and $k = 1$. Consequently, we are compelled to take $r = 1$, $m = 1$ in (5.9).

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