# Asymptotics of integrals of Hermite polynomials

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**Abstract**

Integrals involving products of Hermite polynomials with the weight factor $\exp(-x^2)$ over the interval $(-\infty, \infty)$ are considered. A result of Azor, Gillis and Victor (SIAM J. Math. Anal. **13** (1982) 879–890] is derived by analytic arguments and extended to higher order products. An asymptotic expansion in the case of a product of four Hermite polynomials $H_n(x)$ as $n \to \infty$ is obtained by a discrete analogue of Laplace's method applied to sums.

**Mathematics Subject Classification:**

**Keywords:** Hermite polynomials, Moment integrals, Asymptotic expansion

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## 1. Introduction

Consider an even set of distinguishable elements divided into $k$ subsets containing $n_r$ elements $(r = 1, 2, \ldots, k)$ with $\sum_{r=1}^k n_r$ an even integer, $2s$. In [2], Azor, Gillis and Victor considered the combinatorial problem of the arrangement of these elements into $s$ disjoint pairs. They showed that the number of possible pairings of different types is expressed in terms of integrals of the type

$$I(n_1, n_2, \ldots, n_k) = \int_{-\infty}^{\infty} e^{-x^2} \prod_{r=1}^{k} H_{n_r}(x)\, dx, \tag{1.1}$$

where $H_n(x)$ denotes a Hermite polynomial of degree $n$. The evaluation of the above integral in the case $k = 4$ when $n_1 = n_2$ and $n_3 = n_4$ was obtained by combinatorial arguments and expressed in terms of a terminating ${}_3F_2$ generalised hypergeometric function. In the case $n_r = n$ $(1 \le r \le 4)$ the asymptotics of this integral as $n \to \infty$ were obtained by means of an elaborate argument involving the use of a generating function derived from a contour integral containing a Legendre function. In this paper, we shall obtain the evaluation of (1.1) in the cases $k = 4, 5$ and $6$ by analytic arguments. In addition, we show how an asymptotic expansion in the case $k = 4$ and $n_r = n$ can be obtained as $n \to \infty$ by a discrete analogue of Laplace's method applied to sums.

We assemble here the main properties of the Hermite polynomials that we shall require. The $H_n(x)$ are defined by

$$H_n(x) = (-)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \qquad n = 0, 1, 2, \ldots$$

and have the representation

$$H_n(x) = \begin{cases} (-)^{n/2} \dfrac{n!}{(\frac{1}{2}n)!} \, {}_1F_1(-\frac{1}{2}n; \frac{1}{2}; x^2) & (n \text{ even}), \\[2ex] (-)^{(n-1)/2} \dfrac{n!}{(\frac{1}{2}n - \frac{1}{2})!} 2x \, {}_1F_1(-\frac{1}{2}n + \frac{1}{2}; \frac{3}{2}; x^2) & (n \text{ odd}), \end{cases} \tag{1.2}$$

where ${}_1F_1$ denotes the confluent hypergeometric function. These polynomials satisfy the well-known orthogonality property

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n \sqrt{\pi}\, n!\, \delta_{m,n}, \tag{1.3}$$

where $\delta_{m,n}$ is the Kronecker delta.

From [4, p. 838, 7.376], we have the integrals[1]

$$\int_0^{\infty} e^{-x^2} x^{\nu} H_{2n}(x)\, dx = \frac{(-)^n 2^{2n-1}}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}) \Gamma(n + \tfrac{1}{2}) \, {}_1F_1(-n, \tfrac{1}{2}\nu + \tfrac{1}{2}; \tfrac{1}{2}; 1),$$

$$\int_0^{\infty} e^{-x^2} x^{\nu} H_{2n+1}(x)\, dx = \frac{(-)^n 2^{2n-1}}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}\nu + 1) \Gamma(n + \tfrac{3}{2}) \, {}_1F_1(-n, \tfrac{1}{2}\nu + 1; \tfrac{3}{2}; 1),$$

the first integral holding for $\mathrm{Re}(\nu) > -1$ and the second integral for $\mathrm{Re}(\nu) > -2$. Making use of Vandermonde's theorem [5, p. 243]

$${}_1F_1(-n, r + \tfrac{1}{2}; \tfrac{1}{2}; 1) = \frac{(-r)_n}{(\frac{1}{2})_n},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, we obtain the moment integrals for nonnegative integer $r$

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2r} H_{2n}(x)\, dx = (-)^n 2^{2n} \sqrt{\pi} (\tfrac{1}{2})_r (-r)_n, \tag{1.4}$$

[1] There are misprints in the second of these evaluations in [4].

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2r+1} H_{2n+1}(x)\,dx = (-)^n 2^{2n} \sqrt{\pi} (\tfrac{3}{2})_r (-r)_n. \tag{1.5}$$

Since $(-r)_n = (-)^n r(r-1)\ldots(r-n+1)$, we see that both these moment integrals vanish when $r < n$.

Finally, with $2s = n_1 + n_2 + n_3$, we note the integral (1.1) corresponding to $k = 3$ is given by [4, p. 838, 7.375]

$$\int_{-\infty}^{\infty} e^{-x^2} H_{n_1}(x) H_{n_2}(x) H_{n_3}(x)\,dx = \begin{cases} \dfrac{2^s \sqrt{\pi}\, n_1! n_2! n_3!}{(s-n_1)!(s-n_2)!(s-n_3)!} & (s \text{ even}), \\ 0 & (s \text{ odd}); \end{cases} \tag{1.6}$$

this integral is also clearly zero when $s$ is even if any one of the three indices is greater than the sum of the other two.

## 2. The case $k = 4$

We consider the evaluation of the integral (1.1) when $k = 4$: this requires the sum of the indices $n_r$ to be an even integer for a nonzero value. Thus, the indices can be all even or odd, or two of them can be of different parity. With $n_3 + n_4 = 2p$ even, we can expand the product $H_{n_3}(x)H_{n_4}(x)$ as a linear combination of $H_{2k}(x)$ $(0 \le k \le p)$ in the form

$$H_{n_3}(x) H_{n_4}(x) = \sum_{k=0}^{p} c_{2k} H_{2k}(x), \tag{2.1}$$

where

$$2^{2k} \sqrt{\pi}\, (2k)! c_{2k} = \int_{-\infty}^{\infty} e^{-x^2} H_{n_3}(x) H_{n_4}(x) H_{2k}(x)\,dx$$

by the orthogonality property (1.3). Evaluation of this integral by (1.6) then yields the coefficients $c_{2k}$ in the form

$$c_{2k} = \frac{2^{(n_3+n_4)/2-k}\, n_3! n_4!}{\left(\frac{n_3-n_4}{2}+k\right)! \left(\frac{n_4-n_3}{2}+k\right)! \left(\frac{n_3+n_4}{2}-k\right)!}. \tag{2.2}$$

Then the integral

$$\begin{aligned} I(n_1, \ldots, n_4) &= \sum_{k=0}^{p} c_{2k} \int_{-\infty}^{\infty} e^{-x^2} H_{n_1}(x) H_{n_2}(x) H_{2k}(x)\,dx \\ &= \sum_{k=0}^{p} c_{2k} \frac{2^{(n_1+n_2)/2+k} \sqrt{\pi}\, n_1! n_2! (2k)!}{\left(\frac{n_1-n_2}{2}+k\right)! \left(\frac{n_2-n_1}{2}+k\right)! \left(\frac{n_1+n_2}{2}-k\right)!} \\ &= A \sum_{k=0}^{p} \frac{(-\frac{n_1+n_2}{2})_k (-\frac{n_3+n_4}{2})_k (2k)!}{(\frac{n_1-n_2}{2}+k)!(\frac{n_2-n_1}{2}+k)!(\frac{n_3-n_4}{2}+k)!(\frac{n_4-n_3}{2}+k)!}, \end{aligned}$$

where

$$A = 2^{\sigma/2}\sqrt{\pi}\,\frac{n_1!\,n_2!\,n_3!\,n_4!}{(\frac{n_1+n_2}{2})!(\frac{n_3+n_4}{2})!}, \qquad \sigma = \sum_{r=1}^{r} n_r$$

and we have used the fact that $(-a)_k = (-)^k a!/(a-k)!$. Since $(2k)! = 2^{2k}k!(\frac{1}{2})_k$ by the duplication formula, the above sum can be expressed as a ${}_5F_4$ generalised hypergeometric function to yield the final result

$$I(n_1,\ldots,n_4) = \frac{2^{\sigma/2}\sqrt{\pi}\,n_1!\,n_2!\,n_3!\,n_4!}{(\frac{n_1+n_2}{2})!(\frac{n_3+n_4}{2})!(\frac{n_1-n_2}{2})!(\frac{n_2-n_1}{2})!(\frac{n_3-n_4}{2})!(\frac{n_4-n_3}{2})!}$$

$$\times\,{}_5F_4\left(\begin{matrix} -\frac{n_1+n_2}{2}, -\frac{n_3+n_4}{2}, \frac{1}{2}, 1, 1; \\ \frac{n_1-n_2}{2}+1, \frac{n_2-n_1}{2}+1, \frac{n_3-n_4}{2}+1, \frac{n_4-n_3}{2}+1; \end{matrix}\; 4\right). \qquad (2.3)$$

In the case $n_1 = n_2 = m$, $n_3 = n_4 = n$, the ${}_5F_4$ function reduces to a ${}_3F_2$ function to yield the evaluation

$$\int_{-\infty}^{\infty} e^{-x^2} H_m^2(x) H_n^2(x)\,dx = 2^{m+n}\sqrt{\pi}\,m!n!\,{}_3F_2\left(\begin{matrix} -m, -n, \frac{1}{2}; \\ 1, 1; \end{matrix}\; 4\right) \qquad (2.4)$$

as found in [2] by means of combinatorial arguments. A similar reduction arises when $n_1 = n_2 = n_3 = n$, $n_4 = m$, where $m$ and $n$ have the same parity, to produce

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n^3(x)\,dx = \frac{2^{(m+3n)/2}\sqrt{\pi}(n!)^2 m!}{(\frac{m+n}{2})!}\,F(m,n), \qquad (2.5)$$

where

$$F(m,n) := \frac{1}{(\frac{m-n}{2})!(\frac{n-m}{2})!}\,{}_3F_2\left(\begin{matrix} -n, -\frac{m+n}{2}, \frac{1}{2}; \\ \frac{m-n}{2}+1, \frac{n-m}{2}+1; \end{matrix}\; 4\right). \qquad (2.6)$$

We remark that the above hypergeometric functions have natural cut-offs. For example, the function $F(m,n)$ can be written as

$$F(m,n) = \sum_{k=k_0}^{k_1} \frac{(-n)_k(-\frac{m+n}{2})_k(\frac{1}{2})_k 2^{2k}}{k!(\frac{m-n}{2}+k)!(\frac{n-m}{2}+k)!}, \qquad (2.7)$$

where

$$k_0 = |\tfrac{1}{2}(m-n)|,\ \ k_1 = \min\{n, \tfrac{1}{2}(m+n)\}.$$

From this, it is seen that when $m > 3n$ we have $k_0 > k_1$, so that the sum (2.7) is empty and the integral in (2.5) is therefore zero; when $n > m$ we have $k_0 < k_1$, so that the integral (2.5) is nonzero.

## 3. Examples of (1.1) when $k = 5$ and $k = 6$

The same procedure used in Section 2, combined with the result in (2.5), can be employed to evaluate cases of (1.1) corresponding to $k = 5$ and $k = 6$. Consider the integral (1.1) with $k = 5$ and $n_r = 2n$ $(1 \le r \le 5)$ given by

$$I_n = \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^5(x)\, dx. \tag{3.1}$$

Then, since by (2.1) and (2.2)

$$H_{2n}^2(x) = \sum_{k=0}^{2n} c_{2k} H_{2k}(x), \qquad c_{2k} = \frac{2^{2n-k}((2n)!)^2}{(k!)^2(2n-k)!},$$

we have upon employing (2.5)

$$\begin{aligned} I_n &= \sum_{k=0}^{2n} c_{2k} \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^3(x) H_{2k}(x)\, dx \\ &= 2^{3n}\sqrt{\pi}((2n)!)^2 \sum_{k=0}^{2n} c_{2k} \frac{2^k(2k)!}{(n+k)!} F(2k,2n), \end{aligned}$$

where $F(2k, 2n)$ is defined in (2.6). We therefore have the evaluation

$$I_n = 2^{5n}\sqrt{\pi}\,((2n)!)^4 \sum_{k=0}^{2n} \frac{(\frac{1}{2})_k 2^{2k} F(2k,2n)}{k!(2n-k)!(n+k)!}. \tag{3.2}$$

In a similar manner we can evaluate (1.1) when $k = 6$ and $n_r = n$ $(1 \le r \le 6)$, namely the integral

$$J_n = \int_{-\infty}^{\infty} e^{-x^2} H_n^6(x)\, dx. \tag{3.3}$$

If we first suppose that $n$ is even , we have (upon replacing $n$ by $2n$)

$$H_{2n}^3(x) = \sum_{k=0}^{3n} c_{2k} H_{2k}(x),$$

where by (1.3)

$$2^{2k}\sqrt{\pi}(2k)!c_{2k} = \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^3(x) H_{2k}(x)\, dx. \tag{3.4}$$

It then follows that

$$J_{2n} = \sum_{k=0}^{3n} c_{2k} \int_{-\infty}^{\infty} e^{-x^2} H_{2n}^3(x) H_{2k}(x)\, dx = \sqrt{\pi} \sum_{k=0}^{3n} 2^{2k}(2k)!\, c_{2k}^2.$$

Evaluating the integral in (3.4) by means of (2.5), combined with a similar procedure for odd $n$, we then obtain the result

$$J_n = 2^{3n}\sqrt{\pi}\,((n)!)^4 \begin{cases} \displaystyle\sum_{k=0}^{3n/2} \frac{(\frac{1}{2})_k k! 2^{2k}}{((\frac{1}{2}n+k)!)^2}\, F^2(2k,n) & (n \text{ even}), \\ \displaystyle\sum_{k=0}^{(3n-1)/2} \frac{(\frac{3}{2})_k k! 2^{2k}}{((\frac{1}{2}n+\frac{1}{2}+k)!)^2}\, F^2(2k+1,n) & (n \text{ odd}). \end{cases} \tag{3.5}$$

## 4. Moment integrals

In this section we evaluate the moment integrals involving a product of two Hermite polynomials defined by

$$K(m,n,r) = \int_{-\infty}^{\infty} e^{-x^2} x^r H_m(x) H_n(x)\, dx, \tag{4.1}$$

where $r$ is a nonnegative integer and $m+n+r$ is even.

From (1.2) and (1.4) we find, for even $m$ and $n$,

$$\begin{aligned} K(2m,2n,2r) &= (-)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!(\frac{1}{2})_k} \int_{-\infty}^{\infty} e^{-x^2} x^{2k+2r} H_{2m}(x)\, dx \\ &= (-)^{m+n} 2^{2m}\sqrt{\pi}\, \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!(\frac{1}{2})_k} (\tfrac{1}{2})_{k+r} (-k-r)_m. \end{aligned}$$

Upon noting that

$$(\tfrac{1}{2})_{k+r} = (\tfrac{1}{2})_r (r+\tfrac{1}{2})_k, \qquad (-k-r)_m = (-1)^m \frac{(r+k)!}{(r-m+k)!},$$

we then obtain

$$K(2m,2n,2r) = (-)^n 2^{2m}\sqrt{\pi}\, \frac{(2n)!}{n!} (\tfrac{1}{2})_r \sum_{k=0}^{n} \frac{(-n)_k (r+\frac{1}{2})_k (r+k)!}{k!(\frac{1}{2})_k (r-m+k)!}. \tag{4.2}$$

It is readily seen that when $r < m-n$ $(m > n)$ all terms in the above sum vanish. Since $m$ and $n$ are interchangeable, it then follows that $K(2m,2n,2r) = 0$ when $r < |m-n|$.

An alternative representation is given by expressing the sum in (4.2) as a ${}_3F_2$ hypergeometric function of unit argument to obtain

$$K(2m,2n,2r) = (-)^n 2^{2m}\sqrt{\pi}\, \frac{(2n)!r!}{n!(r-m)!} (\tfrac{1}{2})_r\, {}_3F_2\left(\begin{matrix} -n, r+\frac{1}{2}, r+1; \\ \frac{1}{2}, r-m+1; \end{matrix}\ 1\right). \tag{4.3}$$

A similar procedure can be applied to the remaining two cases of odd $m$, $n$ and $m$, $n$ of different parity with $r$ odd to yield

$$K(2m+1,2n+1,2r) = (-)^n 2^{2m+1}\sqrt{\pi}\frac{(2n+1)!r!}{n!(r-m)!}(\tfrac{3}{2})_r\, {}_3F_2\left(\begin{matrix}-n, r+1, r+\frac{3}{2};\\ \frac{3}{2}, r-m+1;\end{matrix}\ 1\right) \tag{4.4}$$

and

$$K(2m+1,2n,2r+1) = (-)^n 2^{2m}\sqrt{\pi}\frac{(2n)!r!}{n!(r-m)!}(\tfrac{3}{2})_r\, {}_3F_2\left(\begin{matrix}-n, r+1, r+\frac{3}{2};\\ \frac{1}{2}, r-m+1;\end{matrix}\ 1\right). \tag{4.5}$$

It is found that $K(2m+1,2n+1,2r)$ in (4.4) similarly vanishes when $r < |m-n|$, whereas $K(2m+1,2n,2r+1)$ in (4.5) vanishes when $r < m-n$ $(m > n)$ and $r < n-m-1$ $(n > m+1)$.

In the case $m = n$, we define

$$k_r(n) := K(n,n,2r) = \int_{-\infty}^{\infty} e^{-x^2} x^{2r} H_n^2(x)\, dx \tag{4.6}$$

for nonnegative integer $n$. If we apply Thomae's transformation [1, p. 143]

$${}_3F_2\left(\begin{matrix}a, b, c;\\ d, e;\end{matrix}\ 1\right) = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\, {}_3F_2\left(\begin{matrix}d-a, e-a, s;\\ s+b, s+c;\end{matrix}\ 1\right),$$

where in this section $s = d+e-a-b-c$ denotes the parametric excess, to the hypergeometric functions appearing in (4.3) and (4.4) we find that

$$k_r(n) = (-)^r 2^n \sqrt{\pi} n! \begin{cases} (\frac{1}{2})_r\, {}_3F_2\left(\begin{matrix}-r, r+1, \frac{1}{2}n+\frac{1}{2};\\ \frac{1}{2}, 1;\end{matrix}\ 1\right) & (n \text{ even}), \\ (\frac{3}{2})_r\, {}_3F_2\left(\begin{matrix}-r, r+1, \frac{1}{2}n+1;\\ 1, \frac{3}{2};\end{matrix}\ 1\right) & (n \text{ odd}). \end{cases} \tag{4.7}$$

Application of Sheppard's transformation for nonnegative integer $r$ [1, p. 141]

$${}_3F_2\left(\begin{matrix}-r, a, b;\\ d, e;;\end{matrix}\ 1\right) = \frac{(d-a)_r(e-a)_r}{(d)_r(e)_r}\, {}_3F_2\left(\begin{matrix}-r, a, 1-s;\\ a-r+d+1, a-r-e+1;\end{matrix}\ 1\right)$$

shows that the ratio of the two ${}_3F_2$ functions appearing in (4.7) equals $(\frac{3}{2})_r/(\frac{1}{2})_r$. Hence we finally obtain the evaluation

$$k_r(n) = (-)^r 2^{n-r} \sqrt{\pi} n!\, \wp_r(n) \tag{4.8}$$

for nonnegative integers $n$ and $r$, where the polynomial $\wp_r(n)$ is given by

$$\wp_r(n) = 2^r (\tfrac{1}{2})_r\, {}_3F_2\left(\begin{matrix}-r, r+1, \frac{1}{2}n+\frac{1}{2};\\ \frac{1}{2}, 1;\end{matrix}\ 1\right).$$

The first few polynomials $\wp_r(n)$ are then found to be

$$\begin{aligned}
\wp_0(n) &= 1, \qquad \wp_1(n) = 2n+1,\\
\wp_2(n) &= 3(2n^2+2n+1),\\
\wp_3(n) &= 5(4n^3+6n^2+8n+3),\\
\wp_4(n) &= 35(2n^4+4n^3+10n^2+8n+3),\\
\wp_5(n) &= 63(4n^5+10n^4+40n^3+50n^2+46n+15), \;\ldots\;.
\end{aligned}$$

## 5. An asymptotic expansion for $n \to \infty$

The asymptotic behaviour of the integral (1.1) when one or more of the indices $n_r$ is large and $k = 2, 3$ follows immediately from (1.3) and (1.6), respectively. In [2], Azor *et al.* considered the particular case of the next integral in the sequence corresponding $k = 4$ and $n_r = n$ $(1 \leq r \leq n)$, namely

$$I_n = \int_{-\infty}^{\infty} e^{-x^2} H_n^4(x)\, dx,$$

and derived by an elaborate argument an asymptotic estimate of this integral as $n \to \infty$. They expressed $I_n$ as integral involving a Legendre function taken round a contour surrounding the origin in the complex plane and from this constructed a generating function to which they applied Darboux's method. The asymptotics of $I_n$ as $n \to \infty$ were then deduced from the bahaviour of the generating function at its singularities on its circle of convergence.

Here, we shall obtain an asymptotic expansion for $I_n$ by means of the discrete analogue of Laplace's method applied to sums. This method was employed by Stokes [6] in his determination of the leading behaviour of the generalised hypergeometric function ${}_pF_q(x)$ for $x \to +\infty$. An example of the application of this method can also be found in [3, p. 304]. From (2.4) together with $(-n)_k = (-)^k n!/(n-k)!$, we find

$$I_n = 2^{2n}(n!)^4 \sum_{k=0}^{n} \frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3((n-k)!)^2}. \tag{5.1}$$

This sum consists of positive terms which are easily shown to possess a maximum for large $n$ at $k \simeq \frac{2}{3}n \equiv m$. As $n \to \infty$, the terms in the sum (5.1) peak sharply near the maximum term. For arbitrary $\epsilon > 0$ we then have

$$S_n := \sum_{k=0}^{n} \frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3((n-k)!)^2} \;\sim \sum_{k=[m(1-\epsilon)]}^{[m(1+\epsilon)]} \frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3((n-k)!)^2} \tag{5.2}$$

with an error that is subdominant with respect to every power of $1/n$ as $n \to \infty$.

The terms retained in the sum on the right-hand side of (5.2) can now be approximated by means of the well-known expansion for $\Gamma(z)$ given by

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} \sum_{s=0}^{\infty} (-)^s \gamma_s z^{-s}, \qquad z \to +\infty, \tag{5.3}$$

where the first few Stirling coefficients $\gamma_s$ are given by $\gamma_0 = 1$, $\gamma_1 = -\frac{1}{12}$, $\gamma_2 = \frac{1}{288}$, $\gamma_3 = \frac{139}{51840}$. Then, by the duplication formula for the gamma function, we have for large $k$

$$\begin{aligned}\Gamma(k+\tfrac{1}{2}) &= \frac{\sqrt{2\pi}\Gamma(2k)}{2^{2k-\frac{1}{2}}\Gamma(k)} \sim \sqrt{2\pi} k^k e^{-k} \frac{\sum_{s=0}^{\infty}(-)^s \gamma_s (2k)^{-s}}{\sum_{s=0}^{\infty}(-)^s \gamma_s k^{-s}} \\ &\sim \sqrt{2\pi} k^k e^{-k} \left(1 - \frac{1}{24k} + \frac{1}{1152k^2} + \frac{1003}{414720k^3} + \cdots\right) \end{aligned} \tag{5.4}$$

and

$$\Gamma(k+1) = k\Gamma(k) \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} \left(1 + \frac{1}{12k} + \frac{1}{288k^2} - \frac{139}{51840k^3} + \cdots\right). \tag{5.5}$$

From (5.4) and (5.5) we therefore find

$$\frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3} \sim \frac{2^{2k} k^{-2k} e^{2k}}{2\pi k^{3/2}} \left(1 - \frac{7}{24k} + \frac{49}{1152k^2} + \frac{2749}{414720k^3} + \cdots\right)$$

as $k \to +\infty$.

We now set

$$k = m + t, \qquad m = \tfrac{2}{3}n, \qquad \tau = t/m,$$

where $|t|$ is small compared with $m$. We find from (5.3) that

$$(n-k)! = (\tfrac{1}{3}n - t)! \sim \sqrt{2\pi} (\tfrac{1}{3}n - t)^{\frac{1}{3}n - t + \frac{1}{2}} e^{-\frac{1}{3}n + t} \sum_{s=0}^{\infty} \frac{(-)^s 2^s \gamma_s}{m^s (1-2\tau)^s}.$$

Some routine algebra then shows that the terms in the sum on the right-hand side of (5.2) can be written as

$$\begin{aligned}\frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3((n-k)!)^2} &\sim \frac{3}{4\pi^2 n} \left(\frac{3}{2n}\right)^{3/2} \left(\frac{n}{3}\right)^{-2n} e^{2n - 3t^2/m} \\ &\times \frac{(1+\tau)^{-3/2}}{1-2\tau} \frac{\exp\left[3m\tau^2 - 2m(1+\tau)\log(1+\tau)\right]}{\exp\left[m(1-2\tau)\log(1-2\tau)\right]} G(\tau, m),\end{aligned}$$

where

$$G(\tau, m) = \frac{\left(1 - \frac{7(1+\tau)^{-1}}{24m} + \frac{49(1+\tau)^{-2}}{1152m^2} + \frac{2749(1+\tau)^{-3}}{414720m^3} + \cdots\right)}{\left(1 + \frac{(1-2\tau)^{-1}}{6m} + \frac{(1-2\tau)^{-2}}{72m^2} - \frac{139(1-2\tau)^{-3}}{6480m^3} + \cdots\right)^2}.$$

This produces the expansion for large $n$ in the form

$$\frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3((n-k)!)^2} \sim \frac{3}{4\pi^2 n}\left(\frac{3}{2n}\right)^{3/2}\left(\frac{n}{3}\right)^{-2n} e^{2n-3t^2/m}\sum_{s=0}^{\infty} c_s(m)\tau^s,$$

where, omitting the odd coefficients (as these will not be required),

$$c_0(m) = 1-\tfrac{5}{8}m^{-1}+\tfrac{25}{128}m^{-2}+O(m^{-3}), \qquad c_2(m) = \tfrac{23}{6}-\tfrac{231}{64}m^{-1}+O(m^{-2}),$$
$$c_4(m) = -2m+\tfrac{1435}{128}+O(m^{-1}), \qquad c_6(m) = \tfrac{1}{2}m^2+O(1),$$
$$c_8(m) = \tfrac{77}{16}m^2+O(1), \ \ldots \ .$$

We now extend the range of summation in (5.2) to $\pm\infty$ to obtain

$$S_n \sim \frac{3e^{2n}}{4\pi^2 n}\left(\frac{3}{2n}\right)^{3/2}\left(\frac{n}{3}\right)^{-2n}\sum_{t=-\infty}^{\infty} e^{-3t^2/m}\,(c_0(m)+c_1(m)\tau+c_2(m)\tau^2+\cdots). \tag{5.6}$$

The sums in (5.6) may be evaluated using the Poisson-Jacobi transformation [7, p. 124]

$$\sum_{n=-\infty}^{\infty} e^{-an^2} = \sqrt{\frac{\pi}{a}}\left(1+2\sum_{n=1}^{\infty} e^{-\pi^2 n^2/a}\right), \quad \mathrm{Re}(a)>0,$$

so that for $a\to 0+$ we have (neglecting exponentially small terms of order $e^{-\pi^2/a}$)

$$\sum_{n=-\infty}^{\infty} n^{2s}e^{-an^2} \sim (-)^s\left(\frac{d}{da}\right)^s\sqrt{\frac{\pi}{a}} = \Gamma(s+\tfrac{1}{2})\,a^{-s-\frac{1}{2}}$$

for $s=0,1,2,\ldots$ . Since odd powers of $\tau$ yield zero contribution to the sum in (5.6), we then find from (5.1) and (5.6)

$$I_n \sim \frac{3}{4n}\sqrt{\frac{3}{\pi}}\,\frac{n^{-2n}e^{2n}}{2\pi n}\,6^{2n}(n!)^4\sum_{s=0}^{\infty}\frac{c_{2s}(m)\Gamma(s+\frac{1}{2})}{(3m)^s\sqrt{\pi}}.$$

Evaluation of this sum with the above values of $c_{2s}(m)$ $(0\le s\le 3)$ produces the value $1-\frac{5}{12}n^{-1}+O(n^{-2})$. Continuation of this process with the help of *Mathematica* then yields the expansion

$$I_n \sim \frac{3}{4n}\sqrt{\frac{3}{\pi}}\,6^{2n}(n!)^2\left(\sum_{s=0}^{\infty} b_s n^{-s}\right)\left(\sum_{s=0}^{\infty}(-)^s\gamma_s n^{-s}\right)^2,$$

where we have removed a factor of $(n!)^2$ with the help of (5.3) and

$$b_0=1, \quad b_1=-\tfrac{5}{12}, \quad b_2=\tfrac{17}{144}, \quad b_3=\tfrac{287}{12960},$$
$$b_4=\tfrac{1301}{62208}, \quad b_5=\tfrac{1371821}{26127360}, \ \ldots \ .$$

We then finally obtain the expansion[2]

$$I_n \sim \frac{3}{4n}\sqrt{\frac{3}{\pi}}\, 6^{2n}(n!)^2 \sum_{s=0}^{\infty} a_s n^{-s} \qquad (n \to \infty), \tag{5.7}$$

where

$$a_0 = 1, \quad a_1 = -\tfrac{1}{4}, \quad a_2 = \tfrac{1}{16}, \quad a_3 = \tfrac{1}{32},$$
$$a_4 = \tfrac{7}{256}, \quad a_5 = \tfrac{59}{1024}, \ \ldots\ .$$

The first three terms of this expansion were obtained in [2], where the third coefficient was incorrectly given as $a_2 = \frac{3}{16}$. To demonstrate the validity of this expansion we present in Table 1 the absolute relative error in the computation of $I_n$ in (5.1) by means of (5.7) for different values of $n$ and truncation index $s$.

| $s$ | $n = 50$ | $n = 100$ | $n = 200$ |
|---|---|---|---|
| 0 | $5.000 \times 10^{-3}$ | $2.500 \times 10^{-3}$ | $1.250 \times 10^{-3}$ |
| 1 | $2.538 \times 10^{-5}$ | $6.297 \times 10^{-6}$ | $1.568 \times 10^{-6}$ |
| 2 | $2.558 \times 10^{-7}$ | $3.161 \times 10^{-8}$ | $3.928 \times 10^{-9}$ |
| 3 | $4.592 \times 10^{-9}$ | $2.801 \times 10^{-10}$ | $1.729 \times 10^{-11}$ |
| 4 | $1.956 \times 10^{-10}$ | $5.931 \times 10^{-12}$ | $1.827 \times 10^{-13}$ |
| 5 | $1.026 \times 10^{-11}$ | $1.552 \times 10^{-13}$ | $2.387 \times 10^{-15}$ |

Table 1: Values of the absolute relative error in the computation of $I_n$ by the asymptotic expansion (5.7) for different values of $n$ and truncation index $s$.

The integral (1.1) with $n_1 = n_2 = m$, $n_3 = n_4 = n$ is much more straightforward to estimate asymptotically as $n \to \infty$ when $m$ is finite. From (2.5) and (2.7), we have when $n > m$

$$I_{m,n} := \int_{-\infty}^{\infty} e^{-x^2} H_m^2(x) H_n^2(x)\, dx = 2^{m+n}(m!n!)^2 \sum_{k=0}^{m} \frac{\Gamma(k+\frac{1}{2})2^{2k}}{(k!)^3(m-k)!(n-k)!}$$

and, for large $n$, the maximum term in the sum corresponds to $k = m$. With $k = m - j$, we can rewrite the above sum as

$$I_{m,n} = 2^{3m+n}\Gamma(m+\tfrac{1}{2})n! \binom{n}{m} \sum_{j=0}^{m} \frac{d_j(m)}{(n-m+1)_j},$$

[2] An obvious misprint in [2, Eq. (58)] has the factor $(n!)^2$ in the denominator.

where

$$d_j(m) = \frac{(-)^j 2^{-2j} (j!)^2}{(\frac{1}{2}-m)_j} \binom{m}{j}^3 .$$

This is in the form of an inverse factorial expansion in $n$ which is suitable for computation as $n \to \infty$.

The behaviour of the integrals with $n_1 = n_2 = n_3 = m$, $n_4 = n$ and $n_1 = m$, $n_2 = n_3 = n_4 = n$ (where $m$ and $n$ have the same parity) as $n \to \infty$ can be obtained from (2.5) and (2.7). We have

$$\int_{-\infty}^{\infty} e^{-x^2} H_m^3(x) H_n(x)\, dx = 0 \qquad (n > 3m),$$

and

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n^3(x)\, dx = 2^{(m+3n)/2} \sqrt{\pi}\, \frac{(n!)^2 m!}{(\frac{m+n}{2})!}\, F(m,n),$$

where $F(m,n)$ is defined in (2.6). If we suppose $m$ and $n$ are both even and define $(a,j) := (a+j)!(a-j)!$, we can express $F(m,n)$ in the form

$$F(m,n) = \frac{2^n n!}{\sqrt{\pi}}\, (\tfrac{1}{2}m + \tfrac{1}{2}n)! \sum_{j=-m/2}^{m/2} \frac{2^{2j}\Gamma(\frac{1}{2}n + \frac{1}{2} + j)}{(\frac{1}{2}n, j)(\frac{1}{2}m, j)(n - \frac{1}{2}m + j)!};$$

a similar result applies for $m$, $n$ odd. A difficulty arises in the estimation of $F(m,n)$ as $n \to \infty$ since, for finite $m$, the discrete analogue of Laplace's method cannot be employed. Since $(\frac{1}{2}n, j+1)/(\frac{1}{2}n, j) \simeq 1$ as $n \to \infty$, the ratio of consecutive terms in this sum is controlled by $e_j := 2^{2j}\Gamma(\frac{1}{2}n + \frac{1}{2} + j)/(n - \frac{1}{2}m + j)!$. It is then easy to see that $e_{j+1}/e_j \simeq 2$ as $n \to \infty$, so that there is no clear maximum term in the sum.

Finally, we can consider the integral

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) H_{n+m}^2(x)\, dx = 2^{m+2n}(m+n)!n! \sum_{k=0}^{n} \frac{2^{2k}\Gamma(k+\frac{1}{2})}{(k!)^3 (n-k)!(m+n-k)!}$$

by (2.5) and (2.7). As $n \to \infty$, the discrete analogue of Laplace's method can be used to show that that this integral possesses the leading behaviour

$$\frac{3}{4n}\sqrt{\frac{3}{\pi}}\, 6^{2n+m} (m+n)!\, n^{-m} \qquad (n \to \infty).$$

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