The discrete analogue of Laplace's method

R. B. Paris
University of Abertay Dundee, Dundee DD1 1HG, UK

Abstract

We give a justification of the discrete analogue of Laplace's method applied to the asymptotic estimation of sums consisting of positive terms. The case considered is the series related to the hypergeometric function ${}_pF_{q-1}(x)$ (with $q \geq p+1$) as $x \to +\infty$ discussed by Stokes [Proc. Camb. Phil. Soc. **6** (1889) 362–366]. Two examples are given in which it is shown how higher order terms in the asymptotic expansion may be derived by this procedure.

Keywords: Laplace's method, Sums, Asymptotic expansion

---

1. Introduction

Laplace's approximation is one of the most fundamental asymptotic techniques for the estimation of integrals containing a large parameter or variable. For integrals of the form

$$I(x) = \int_a^b f(t)e^{x\psi(t)}dt \qquad (x \to +\infty),$$

where $f(t)$ and $\psi(t)$ are real continuous functions defined on the interval $[a, b]$ (which may be infinite), the dominant contribution as $x \to +\infty$ arises from a neighbourhood of the point where $\psi(t)$ attains its maximum value. When $\psi(t)$ possesses a single maximum at the point $t_0 \in (a, b)$, so that $\psi'(t_0) = 0$, $\psi''(t_0) < 0$ and $f(t_0) \neq 0$, then $I(x)$ has the asymptotic behaviour

$$I(x) \sim f(t_0)e^{x\psi(t_0)} \left(\frac{-2\pi}{x\psi''(t_0)}\right)^{1/2} \qquad (x \to +\infty);$$

see, for example, [4, p. 39] or [16, p. 57].

The same principle may also be applied to the sum of a series of positive terms, in which the terms steadily increase up to a certain point and then steadily decrease. The asymptotic behaviour of the sum of the series can then be obtained by a discrete analogue of Laplace's method by consideration of the order of magnitude of the greatest term in the series. In 1889, Stokes [12] published a short paper in which he applied this principle to obtain the leading asymptotic behaviour of the hypergeometric-type series

$$F(x) = \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} \Gamma(n + a_r)}{\prod_{r=1}^{q} \Gamma(n + b_r)}\, x^n \qquad (q \geq p+1,\ |x| < \infty), \tag{1.1}$$

where $p \geq 0$, $q \geq 1$ are integers, a_r $(1 \leq r \leq p)$ and b_r $(1 \leq r \leq q)$ are positive parameters and $x > 0$. The function $F(x)$ covers many cases of important special functions in physical applications and, when $b_q = 1$ say, is proportional to the generalised hypergeometric function ${}_pF_{q-1}(x)$ with numeratorial parameters a_r $(1 \leq r \leq p)$ and denominatorial parameters b_r $(1 \leq r \leq q-1)$. Stokes argued that the dominant contribution to $F(x)$ as $x \to \infty$ arose from the terms in the series situated in the neighbourhood of the greatest term. By approximation of these terms in the form of a Gaussian exponential followed by replacement of the sum by an integral with limits extended to $\pm\infty$, Stokes showed in a non-rigorous fashion that

$$F(x) \sim (2\pi)^{(1-\kappa)/2}\kappa^{-1/2}x^{(\vartheta+\frac{1}{2})/\kappa}\exp(\kappa x^{1/\kappa}) \qquad (x\to+\infty), \tag{1.2}$$

where

$$\kappa = q-p, \qquad \vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \tfrac{1}{2}\kappa. \tag{1.3}$$

This result appears to be the earliest attempt at the determination of the asymptotic behaviour of a series of the form (1.1). An application of this principle that the large-argument behaviour of a series is controlled by the magnitude of the greatest term was made by Hardy [5] in the determination of the zeros of a class of integral functions.

Relatively little use appears to have been made of the discrete analogue of Laplace's method, presumably on account of its being confined to series of positive terms and the heuristic nature of its arguments. Examples can be found in [14, p. 8] and in the book by Bender & Orszag [2, p. 304], where they derive the leading asymptotic behaviour of the sum

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^\alpha} \qquad (\alpha>0) \tag{1.4}$$

as $x\to+\infty$. In the case of integer α, this latter function is a particular case of $F(x)$ in (1.1) with $p=0$, $q=\alpha$. The same example has been discussed in Olver's book [8, p. 307] but using a different approach based on an integral representation of the sum together with Laplace's approximation for integrals.[1] A recent proof of a discrete analogue of Laplace's method applied to sums of the form $\sum_{k=0}^{n} f_n(k)q^{g_n(k)}$ as $n\to+\infty$, where $f_n(k)$ and $g_n(k)$ are functions defined on nonnegative integers and $0<q<1$, has been given in [13]. These authors applied their results to derive asymptotic formulas for the q^{-1}-Hermite, the Stieltjes-Wigert and the q-Laguerre polynomials.

In this paper, we present a justification of Stokes' arguments for the discrete analogue of Laplace's method applied to the function $F(x)$ in (1.1). Although other more general methods are available for the asymptotics of $F(x)$ when x is a large complex variable, namely the classical Laplace method applied to an integral representation of the sum in (1.1) or the asymptotic theory of hypergeometric-type functions developed in [3, 17], our aim here is to put Stokes' arguments on a more rigorous foundation. We shall restrict our attention to the case of positive parameters and variable x considered by Stokes, although it may be possible to extend the arguments to cover the case of complex x by using the ideas given in [6] applied to the determination of the relation of the maximum modulus of an integral function to its maximum term. In addition, we shall show how higher order terms in the expansion of $F(x)$ as $x\to+\infty$ can also be derived.

The results obtained are then used to give expansions for two functions. The first example is the sum defined in (1.4) and the second example comes from a problem in combinatorics expressed in the form of an integral of a product of Hermite polynomials over $(-\infty,\infty)$.

2. Preliminary lemmas

We first state and prove two lemmas that will be required in the asymptotic discussion of $F(x)$. In the course of our analysis it is necessary to introduce the positive parameter $\epsilon<1$ that will be chosen to scale with the asymptotic variable x given by

$$\epsilon \sim x^{-\nu}, \quad \tfrac{1}{3}<\nu<\tfrac{1}{2} \qquad (x\to+\infty). \tag{2.1}$$

[1] Olver's analysis is restricted to the case $0<\alpha\le 4$.

Lemma 1 *Let $a = \kappa/(2x)$, $0 \le \delta < 1$ and $\mu \sim \epsilon x$, where κ is defined in (1.3), $x > 0$ and ϵ is specified in (2.1). Then for nonnegative integer r, we have*

$$S_{\pm} \equiv \sum_{k=\mu+1}^{\infty} (k \pm \delta)^r e^{-a(k\pm\delta)^2} = O(x^{r+\nu} e^{-a\mu^2}) \qquad (r = 0, 1, 2, \ldots) \tag{2.2}$$

as $x \to \infty$ ($a \to 0+$).

Proof Consider first the sum

$$S_+ = \sum_{k=\mu+1}^{\infty} (k+\delta)^r e^{-a(k+\delta)^2} < \sum_{k=\mu}^{\infty} (k+1)^r e^{-ak^2} \tag{2.3}$$

for $a > 0$ and $0 \le \delta < 1$. Then

$$\begin{aligned} \sum_{k=\mu}^{\infty} (k+1)^r e^{-ak^2} &< e^{-a\mu^2} (\mu+1)^r \sum_{k=0}^{\infty} \left(\frac{\mu+k+1}{\mu+1} \right)^r e^{-2ka\mu} \\ &< e^{-a\mu^2} (\mu+1)^r \left\{ 1 + \sum_{k=1}^{\infty} (2k)^r e^{-2ka\mu} \right\}. \end{aligned}$$

The sum in braces can be expressed as $(-)^r \Upsilon_r(a\mu)$, where the function $\Upsilon_r(z)$ is defined by

$$\Upsilon_r(z) := \sum_{k=1}^{\infty} (-2k)^r e^{-2kz} = \left(\frac{d}{dz} \right)^r (1 - e^{-2z})^{-1}.$$

Routine calculations show that $\Upsilon_r(z) = O(z^{-r-1})$ as $z \to 0+$ and, since $a\mu \sim \frac{1}{2}\kappa x^{-\nu}$, it then follows that

$$S_+ = O((\epsilon x)^r x^{(r+1)\nu} e^{-a\mu^2}) = O(x^{r+\nu} e^{-a\mu^2}). \tag{2.4}$$

With $a\mu^2 = \frac{1}{2}\kappa\epsilon^2 x \sim \frac{1}{2}\kappa x^{1-2\nu}$, the exponential factor in this estimate is therefore seen to be exponentially small as $x \to \infty$ provided $\nu < \frac{1}{2}$.

In a similar manner, we obtain for the second sum

$$S_- = \sum_{k=\mu+1}^{\infty} (k-\delta)^r e^{-a(k-\delta)^2} < \sum_{k=\mu+1}^{\infty} k^r e^{-a(k-1)^2} = \sum_{k=\mu}^{\infty} (k+1)^r e^{-ak^2}.$$

From (2.3) and (2.4), it follows that S_- is also $O(x^{r+\nu} e^{-a\mu^2})$ as $x \to \infty$. □

Lemma 2 *Let $a > 0$, $0 \le \delta < 1$ and r be a nonnegative integer. Then, upon neglecting exponentially small terms, we have*

$$\sum_{k=-\infty}^{\infty} (k+\delta)^{2r} e^{-a(k+\delta)^2} \sim \sqrt{\frac{\pi}{a}}\, a^{-r} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \tag{2.5}$$

and

$$\sum_{k=-\infty}^{\infty} (k+\delta)^{2r+1} e^{-a(k+\delta)^2} \sim 0 \tag{2.6}$$

as $a \to 0$.

Proof By the Poisson-Jacobi transformation [15, p. 124] we have for $a > 0$ and $0 \le \delta < 1$

$$S(a,\delta) := \sum_{k=-\infty}^{\infty} e^{-a(k+\delta)^2} = \sqrt{\frac{\pi}{a}} + \Psi(a,\delta), \qquad \Psi(a,\delta) := 2\sqrt{\frac{\pi}{a}} \sum_{n=1}^{\infty} e^{-\pi^2 n^2/a} \cos(2\pi n\delta).$$

As $a \to 0+$, the function $\Psi(a,\delta)$ and its partial derivatives with respect to a and δ are all exponentially small controlled by $e^{-\pi^2/a}$.

Then we have

$$\frac{\partial^r S}{\partial a^r} = (-)^r \sum_{k=-\infty}^{\infty} (k+\delta)^{2r} e^{-a(k+\delta)^2} = (-)^r \sqrt{\frac{\pi}{a}}\, a^{-r}\, \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} + \frac{\partial^r \Psi}{\partial a^r}.$$

Since $\partial^r\Psi/\partial a^r$ is exponentially small for $r = 1, 2, \ldots$ as $a \to 0+$, the result in (2.5) follows.

To deal with the sum in (2.6), we first note that

$$\frac{\partial S}{\partial \delta} = -2a \sum_{k=-\infty}^{\infty} (k+\delta) e^{-a(k+\delta)^2} = \frac{\partial \Psi}{\partial \delta},$$

so that

$$\frac{\partial^{r+1} S}{\partial a^r \partial \delta} = (-)^{r+1} 2a \sum_{k=-\infty}^{\infty} (k+\delta)^{2r+1} e^{-a(k+\delta)^2} = \frac{\partial^{r+1}\Psi}{\partial a^r \partial \delta}.$$

As all partial derivatives of $\Psi(a,\delta)$ are exponentially small in the limit $a \to 0+$, the result in (2.6) follows. □

3. The dominant contribution to $F(x)$

For convenience in presentation we replace the variable x in (1.1) by x^κ and consider the function

$$F(x) = \sum_{n=0}^{\infty} u_n, \qquad u_n := \frac{\prod_{r=1}^{p} \Gamma(n+a_r)}{\prod_{r=1}^{q} \Gamma(n+b_r)}\, x^{\kappa n}, \tag{3.1}$$

where $\kappa = q - p \geq 1$, a_r $(1 \leq r \leq p)$ and b_r $(1 \leq r \leq q)$ are positive parameters and $x > 0$. The maximum term in this series can be obtained by examination of the ratio of the $(n+1)$th term to the nth term to obtain

$$\frac{u_{n+1}}{u_n} = \frac{\prod_{r=1}^{p}(n+a_r)}{\prod_{r=1}^{q}(n+b_r)}\, x^\kappa = \left(\frac{x}{n}\right)^\kappa \frac{\prod_{r=1}^{p}(1+a_r/n)}{\prod_{r=1}^{q}(1+b_r/n)}.$$

It then follows that the greatest term in the series corresponds to $n \simeq x$; for large x, the terms u_n increase monotonically until $n \simeq x$ and thereafter decrease monotonically with increasing n.

We let $N = \lceil x \rceil$, $x = \lceil x \rceil - \delta$, with $0 \leq \delta < 1$, and with ϵ as specified in (2.1) define the integers $n_\pm$ by

$$n_+ = \lceil (1+\epsilon)N \rceil, \qquad n_- = \lfloor (1-\epsilon)N \rfloor.$$

Then we can write

$$F(x) = \sum_{n=n_-}^{n_+} u_n + \sum_{n<n_-} u_n + \sum_{n>n_+} u_n. \tag{3.2}$$

The dominant contribution to $F(x)$ as $x \to \infty$ arises from the terms with $n_- \leq n \leq n_+$, where the series is sharply peaked near $n = N$, and will have a value controlled by the maximum term u_N. The contributions from the tails $n < n_-$ and $n > n_+$ are then exponentially small compared to u_N, as we now demonstrate.

We have

$$\frac{1}{u_N} \sum_{n<n_-} u_n < \frac{1}{u_N} \sum_{n \leq n_-} u_n < \frac{u_{n_-}}{u_N}(1+n_-) \tag{3.3}$$

due to the monotonic nature of u_n for $n \leq n_-$, and

$$\begin{aligned}\frac{1}{u_N} \sum_{n>n_+} u_n \;&<\; \frac{u_{n_+}}{u_N} \sum_{s=0}^{\infty} \frac{u_{s+n_+}}{u_{n_+}} = \frac{u_{n_+}}{u_N} \sum_{s=0}^{\infty} \frac{\prod_{r=1}^{p}(n_+ + a_r)_s}{\prod_{r=1}^{q}(n_+ + b_r)_s}\, x^{\kappa s} \\ &= \frac{u_{n_+}}{u_N}\, {}_{p+1}F_q\left(\begin{matrix} 1,\; n_+ + a_1, \ldots, n_+ + a_p \\ n_+ + b_1, \ldots, n_+ + b_q \end{matrix}\,\middle|\, x^\kappa\right),\end{aligned} \tag{3.4}$$

where $(a)_s = \Gamma(a+s)/\Gamma(a)$ is Pochhammer's symbol and ${}_{p+1}F_q$ denotes the generalised hypergeometric function with $p+1$ numeratorial parameters and q denominatorial parameters.

Since $n_+ \sim (1+\epsilon)x$, the above hypergeometric function is 'balanced' in the sense that the sth term in its series expansion as $x \to \infty$ is $O(1)$ and approximated by $n_+^{-\kappa s} x^{\kappa s} \sim (1+\epsilon)^{-\kappa s}$. By the confluence principle [7, §3.5], we then find that

$$ {}_{p+1}F_q \left(\begin{array}{c} 1,\ n_+ + a_1, \ldots, n_+ + a_p \\ n_+ + b_1, \ldots, n_+ + b_q \end{array} \middle| x^\kappa \right) \sim \sum_{s=0}^{\infty} (1+\epsilon)^{-\kappa s} = \frac{1}{1-(1+\epsilon)^{-\kappa}} = O(x^\nu) \tag{3.5} $$

as $x \to \infty$. With $n_\pm \sim (1 \pm \epsilon)x$, $N \sim x$, Stirling's formula shows that

$$ \frac{\Gamma(n_\pm + \alpha)}{\Gamma(N+\alpha)} \sim e^{N-n_\pm} \frac{(n_\pm + \alpha)^{n_\pm + \alpha - \frac{1}{2}}}{(N+\alpha)^{N+\alpha-\frac{1}{2}}} = O(e^{\mp \epsilon x} x^{n_\pm - N} (1 \pm \epsilon)^{n_\pm}) $$

for large x and finite α. Then

$$ \frac{u_{n_\pm}}{u_N} = x^{\kappa(n_\pm - N)} O(e^{\pm \kappa \epsilon x} x^{\kappa(N-n_\pm)} (1 \pm \epsilon)^{-\kappa n_\pm}) = O(e^{-\kappa x \Lambda_\pm(\epsilon)}), $$

where

$$ \Lambda_\pm(u) := (1 \pm u)\log(1 \pm u) \mp u. $$

It is easily shown, when $0 < u < 1$, that $\Lambda_+(u) > \frac{1}{2}u^2(1-u)$ and $\Lambda_-(u) > \frac{1}{2}u^2$. Hence the ratios $u_{n_\pm}/u_N = O(\exp(-\kappa x^{1-2\nu}))$, which are exponentially small as $x \to \infty$ since $\nu < \frac{1}{2}$. It then follows from the bounds in (3.3), (3.4) and (3.5) that the tails of the series compared to the maximum term u_N are also exponentially small as $x \to \infty$. The dominant contributuion to $F(x)$ therefore arises from the first sum in (3.2) taken over $n_- \le n \le n_+$.

4. Derivation of the expansion for $F(x)$

From Section 3, the function $F(x)$ can be written as

$$ \begin{aligned} F(x) &\sim \sum_{n=n_-}^{n_+} u_n = \sum_{k=-\mu}^{\mu} u_{k+N} \\ &= \sum_{k=-\mu}^{\mu} \frac{\prod_{r=1}^{p} \Gamma(\sigma + a_r)}{\prod_{r=1}^{q} \Gamma(\sigma + b_r)} x^{\kappa\sigma}, \qquad \sigma \equiv \sigma(k) = x + k + \delta, \end{aligned} \tag{4.1} $$

to within an exponentially small error as $x \to \infty$, where we have used $N = x + \delta$ and the fact that[2] $n_\pm - N = \pm\mu$, with $\mu = \lceil \epsilon N \rceil$. The terms in the sum (4.1) can now be approximated for large x by means of the well-known Stirling expansion for $\Gamma(z)$ given by

$$ \Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \sum_{s=0}^{\infty} (-)^s \gamma_s z^{-s} \qquad (z \to +\infty), \tag{4.2} $$

where the first few Stirling coefficients γ_s have the values $\gamma_0 = 1$, $\gamma_1 = -\frac{1}{12}$, $\gamma_2 = \frac{1}{288}$, $\gamma_3 = \frac{139}{51840}$. Then after some routine algebra we obtain

$$ \Gamma(z+a) \sim \sqrt{2\pi}\, z^{z+a-\frac{1}{2}} e^{-z} \left(1 + \frac{B_1(a)}{z} + \frac{B_2(a)}{z^2} + \cdots \right) \tag{4.3} $$

as $z \to +\infty$, where

$$ B_1(a) = \tfrac{1}{2}a(a-1) + \tfrac{1}{12}, \qquad B_2(a) = \tfrac{5}{12}a^2(1-a) + \tfrac{1}{8}a(a^3-1) + \tfrac{1}{288}, \ldots. $$

[2] For integer N and $x > 0$, we have $\lceil N + x \rceil = N + \lceil x \rceil$ and $\lfloor N - x \rfloor = N - \lceil x \rceil$.

4.1. **Expansion of u_n for $n \simeq N$**

Use of the expansion (4.3) then shows that for $\sigma \to \infty$

$$\frac{\prod_{r=1}^{p} \Gamma(\sigma + a_r)}{\prod_{r=1}^{q} \Gamma(\sigma + b_r)} \sim (2\pi)^{-\kappa/2} \sigma^{-\kappa\sigma+\vartheta} e^{\kappa\sigma} \frac{\prod_{r=1}^{p} \left(1 + \frac{B_1(a_r)}{\sigma} + \frac{B_2(a_r)}{\sigma^2} + \cdots\right)}{\prod_{r=1}^{q} \left(1 + \frac{B_1(b_r)}{\sigma} + \frac{B_2(b_r)}{\sigma^2} + \cdots\right)}$$

$$= (2\pi)^{-\kappa/2} \sigma^{-\kappa\sigma+\vartheta} e^{\kappa\sigma} \sum_{s=0}^{\infty} D_s \sigma^{-s},$$

where the parameters κ, ϑ are defined in (1.3) and the first few coefficients D_s are given by

$$D_0 = 1, \qquad D_1 = \sum_{r=1}^{p} B_1(a_r) - \sum_{r=1}^{q} B_1(b_r),$$

$$D_2 = \sum_{r=1}^{p} B_1(a_r) - \sum_{r=1}^{q} B_2(b_r) + \sum_{r=1}^{p-1} \sum_{j=r+1}^{p} B_1(a_r) B_1(a_j) - \sum_{r=1}^{q-1} \sum_{j=r+1}^{q} B_1(b_r) B_1(b_j)$$
$$+ \left(\sum_{r=1}^{q} B_1(b_r)\right)^2 - \sum_{r=1}^{p} B_1(a_r) \sum_{r=1}^{q} B_1(b_r), \; \ldots \; . \tag{4.4}$$

Now put $\sigma = x(1 + u)$, where $u = (k + \delta)/x$ and define the variable

$$\tau_k = \frac{k + \delta}{x^{1/2}} \qquad (-\mu \le k \le \mu). \tag{4.5}$$

Then we find

$$\frac{\prod_{r=1}^{p} \Gamma(\sigma + a_r)}{\prod_{r=1}^{q} \Gamma(\sigma + b_r)} x^{\kappa\sigma} \sim (2\pi)^{-\kappa/2} x^{\vartheta} e^{\kappa\sigma} (1 + u)^{\vartheta - \kappa\sigma} \sum_{s=0}^{\infty} \frac{D_s}{x^s} (1 + u)^{-s},$$

where

$$e^{\kappa\sigma} (1 + u)^{-\kappa\sigma} = e^{\kappa x(1+u)\{1 - \log(1+u)\}} = e^{\kappa x - \frac{1}{2}\kappa\tau_k^2} e^{\kappa T},$$

$$T = \frac{\tau_k^3}{6x^{1/2}} - \frac{\tau_k^4}{12x} + \frac{\tau_k^5}{20x^{3/2}} - \frac{\tau_k^6}{30x^2} + \cdots \; .$$

For $|k| \le \mu$, we have

$$\frac{|\tau_{\pm\mu}|^{s+3}}{x^{(s+1)/2}} = O(x^{1-(s+3)\nu}) \qquad (s = 0, 1, 2, \ldots)$$

and $|u| = O(x^{-\nu})$. Since $\nu > \frac{1}{3}$, it follows that each term appearing in T (with $|k| \le \mu$) is $o(x^{-s\nu})$ as $x \to \infty$ and so, upon application of the binomial theorem, we obtain

$$e^{\kappa T} \sum_{s=0}^{\infty} \frac{D_s}{x^s} (1 + u)^{\vartheta - s} = \sum_{s=0}^{\infty} \frac{P_s(\tau_k)}{x^{s/2}},$$

where

$$P_0(\tau_k) = 1, \qquad P_1(\tau_k) = \vartheta\tau_k + \tfrac{1}{6}\kappa\tau_k^3.$$

The $P_s(\tau_k)$ are polynomials in τ_k of degree $3s$ and consist of even (resp. odd) powers of τ_k according as s is even (resp. odd), and have the form

$$P_s(\tau_k) = \sum_{r=0}^{\lfloor 3s/2 \rfloor} \beta_r^{(s)} \tau_k^{2r+\omega}, \tag{4.6}$$

where $\omega = 0$ or 1 according as s is even or odd, respectively. For $s = 2, 3$ and 4, the coefficients $\beta_r^{(s)}$ are given by

$$\beta_0^{(2)} = D_1, \quad \beta_1^{(2)} = \tfrac{1}{2}\vartheta(\vartheta - 1), \quad \beta_2^{(2)} = \tfrac{1}{6}\kappa(\vartheta - \tfrac{1}{2}), \quad \beta_3^{(2)} = \tfrac{1}{72}\kappa^2, \tag{4.7}$$

$$\beta_0^{(3)} = D_1(\vartheta - 1), \quad \beta_1^{(3)} = \tfrac{1}{6}\{\vartheta(\vartheta - 1)(\vartheta - 2) + \kappa D_1\}, \quad \beta_2^{(3)} = \tfrac{1}{60}\kappa(3 - 10\vartheta + 5\vartheta^2),$$

$$\beta_3^{(3)} = \tfrac{1}{72}\kappa^2(\vartheta - 1), \quad \beta_4^{(3)} = \tfrac{1}{1296}\kappa^3,$$

and

$$\beta_0^{(4)} = D_2, \quad \beta_1^{(4)} = \tfrac{1}{2}D_1(\vartheta - 1)(\vartheta - 2), \quad \beta_2^{(4)} = \tfrac{1}{24}\{\vartheta(\vartheta - 1)(\vartheta - 2)(\vartheta - 3) + 2\kappa D_1(2\vartheta - 3)\},$$

$$\beta_3^{(4)} = \tfrac{1}{360}\kappa(2\vartheta - 3)(4 - 15\vartheta + 5\vartheta^2) + \tfrac{1}{72}\kappa^2 D_1, \quad \beta_4^{(4)} = \tfrac{1}{1440}\kappa^2(17 - 30\vartheta + 10\vartheta^2),$$

$$\beta_5^{(4)} = \tfrac{1}{2592}\kappa^3(2\vartheta - 3), \quad \beta_6^{(4)} = \tfrac{1}{31104}\kappa^4. \tag{4.8}$$

The expansion of the summand in (4.1) in the neighbourhood of the greatest term $n = N$ then finally becomes

$$\frac{\prod_{r=1}^p \Gamma(\sigma + a_r)}{\prod_{r=1}^q \Gamma(\sigma + b_r)} x^{\kappa\sigma} \sim (2\pi)^{-\kappa/2} x^\vartheta e^{\kappa x - \frac{1}{2}\kappa\tau_k^2} \sum_{s=0}^\infty \frac{P_s(\tau_k)}{x^{s/2}}, \qquad (x \to \infty) \tag{4.9}$$

for $|k| \leq \mu$, where τ_k is defined in (4.5), $\mu \sim \epsilon x$ and ϵ is specified in (2.1).

4.2. **The asymptotic evaluation of the dominant sum**

From (4.1) and (4.9), the dominant contribution to $F(x)$ can accordingly be expressed in the form

$$F(x) \sim (2\pi)^{-\kappa/2} x^\vartheta e^{\kappa x} \sum_{k=-\mu}^{\mu} e^{-\frac{1}{2}\kappa\tau_k^2} \left(1 + \frac{P_1(\tau_k)}{x^{1/2}} + \frac{P_2(\tau_k)}{x} + \cdots\right). \tag{4.10}$$

Now

$$\sum_{k=\mu+1}^{\infty} \tau_k^r e^{-\frac{1}{2}\kappa\tau_k^2} = x^{-r/2} S_+, \qquad \sum_{k=-\mu-1}^{-\infty} \tau_k^r e^{-\frac{1}{2}\kappa\tau_k^2} = x^{-r/2} S_-$$

for nonnegative integer r, where $S_\pm$ are defined in (2.2). Then application of Lemma 1 shows that both these sums are $O(x^{r/2+\nu} e^{-a\mu^2})$ as $x \to \infty$. Consequently, since $a\mu^2 = \frac{1}{2}\kappa\epsilon^2 x \sim x^{1-2\nu}$ with $\nu < \frac{1}{2}$, the limits of summation in (4.10) can be extended to $\pm\infty$, with the introduction of additional exponentially small errors as $x \to \infty$ resulting from each power τ_k^r present in $P_s(\tau_k)$. Thus we find

$$F(x) \sim (2\pi)^{-\kappa/2} x^\vartheta e^{\kappa x} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{2}\kappa\tau_k^2} \left\{1 + \frac{P_1(\tau_k)}{x^{1/2}} + \frac{P_2(\tau_k)}{x} + \cdots\right\}. \tag{4.11}$$

The terms in (4.11) can be evaluated by Lemma 2. From (2.5) with $a = \kappa/(2x)$, we have upon neglecting exponentially small terms

$$\sum_{k=-\infty}^{\infty} \tau_k^{2r} e^{-\frac{1}{2}\kappa\tau_k^2} \sim \left(\frac{2\pi x}{\kappa}\right)^{1/2} (\tfrac{1}{2}\kappa)^{-r} \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})} \qquad (r = 0, 1, 2, \ldots) \tag{4.12}$$

as $x \to \infty$, with the analogous sum involving odd powers of τ_k being exponentially small by (2.6). Then, from (4.6), it follows that

$$\sum_{k=-\infty}^{\infty} e^{-\frac{1}{2}\kappa\tau_k^2} P_{2s}(\tau_k) = \sum_{r=0}^{3s} \beta_r^{(2s)} \sum_{k=-\infty}^{\infty} \tau_k^{2r} e^{-\frac{1}{2}\kappa\tau_k^2}$$
$$\sim \left(\frac{2\pi x}{\kappa}\right)^{1/2} \sum_{r=0}^{3s} \beta_r^{(2s)} (\tfrac{1}{2}\kappa)^{-r} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

as $x \to \infty$, neglecting exponentially small terms. The contribution to the sum from the odd coefficients $P_{2s+1}(\tau_k)$ in (4.11) is exponentially small in the limit $x \to \infty$.

If we now define the coefficients c_s by

$$c_s = \kappa^s \sum_{r=0}^{3s} \beta_r^{(2s)} (\tfrac{1}{2}\kappa)^{-r} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}, \tag{4.13}$$

the expansion for $F(x)$ finally takes the form

$$F(x) \sim (2\pi)^{(1-\kappa)/2} \kappa^{-1/2} x^{\vartheta+\frac{1}{2}} e^{\kappa x} \sum_{s=0}^{\infty} \frac{c_s}{(\kappa x)^s} \qquad (x \to \infty), \tag{4.14}$$

where $c_0 = 1$. The next coefficient c_1 can be evaluated by noting, from (4.4), that

$$D_1 = \frac{1}{2}\sum_{r=1}^{p} a_r(a_r-1) - \frac{1}{2}\sum_{r=1}^{q} b_r(b_r-1) - \frac{\kappa}{12}$$

to find

$$c_1 = \kappa D_1 + \tfrac{1}{2}\vartheta(\vartheta-1) + \tfrac{1}{2}(\vartheta-\tfrac{1}{2}) + \tfrac{5}{24}$$
$$= \tfrac{1}{2}\kappa \left\{\sum_{r=1}^{p} a_r(a_r-1) - \sum_{r=1}^{q} b_r(b_r-1) + \frac{\vartheta^2}{\kappa}\right\} - \frac{\kappa^2}{12} - \frac{1}{24}. \tag{4.15}$$

The coefficient c_1 agrees with that given in [9, Appendix A] when due account is taken of the different definition of $F(x)$. The expression for the coefficient c_2 is not stated due to its complexity but can be obtained from (4.8), (4.13) and evaluation of D_2 in (4.4). Higher coefficients can be derived with the aid of *Mathematica* when dealing with specific cases, as is carried out in Section 5. When the variable x is replaced by $x^{1/\kappa}$, the leading form of (4.14) agrees with Stokes' result given in (1.2). This concludes the justification of the discrete analogue of Laplace's method applied to the sum in (3.1).

A more detailed analysis of $F(x)$ [17, 3] (see also [11, §2.3]) shows that the exponential expansion in (4.14) holds for complex values of x in a sector enclosing the positive real axis. When $\kappa > 2$, additional subdominant exponential expansions appear in the asymptotics of $F(x)$. In addition, when $p \neq 0$, there is a subdominant algebraic expansion present, which undergoes a Stokes phenomenon on the positive real x-axis.

5. Examples

We conclude this paper by giving two examples of the discrete analogue of Laplace's method applied to sums where higher order terms in the expansion are obtained.

Example 1. Our first example is the function defined in (1.4), which we rewrite in the form

$$\psi(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{(n!)^{\alpha}}$$

For positive integer α, this function is a particular case of $F(x)$ in (3.1) corresponding to $p = 0$, $q = \alpha$ with $b_r = 1$ $(1 \le r \le q)$, for which the parameters in (1.3) have the values $\kappa = \alpha$ and $\vartheta = -\frac{1}{2}\alpha$. It is easily verified that the maximum term in the series corresponds to $n \simeq x$.

Then, with $N = \lceil x \rceil$, $x = \lceil x \rceil - \delta$, $0 \le \delta < 1$, $\sigma = x + k + \delta$ and τ_k as defined in (4.5), we have from (4.9)

$$\frac{x^{\alpha\sigma}}{(\sigma!)^\alpha} \sim \frac{e^{\alpha x - \frac{1}{2}\alpha\tau_k^2}}{(2\pi x)^{\alpha/2}} \sum_{s=0}^\infty \frac{P_s(\tau_k)}{x^{s/2}} \tag{5.1}$$

for $x \to \infty$ and $|k| \le \mu$. From (4.4), we find $D_1 = -\alpha/12$ and $D_2 = \alpha^2/288$, which may then be substituted into the coefficients $\beta_r^{(s)}$ in (4.7) and (4.8) to yield the coefficients c_s $(s \le 2)$ in (4.13) and (4.15). Then from (4.14), we obtain the expansion

$$\psi(x) \sim \frac{\alpha^{-1/2} e^{\alpha x}}{(2\pi x)^{(\alpha-1)/2}} \sum_{s=0}^\infty \frac{c_s}{(\alpha x)^s} \qquad (x \to \infty), \tag{5.2}$$

where

$$c_0 = 1, \quad c_1 = \tfrac{1}{24}(\alpha^2 - 1), \quad c_2 = \tfrac{1}{1152}(\alpha^4 + 22\alpha^2 - 23).$$

Continuation of the expansion process in (4.9) and (4.10) with the aid of *Mathematica* yields the additional coefficients

$$c_3 = \tfrac{1}{414720}(5\alpha^6 - 303\alpha^4 + 11535\alpha^2 - 11237),$$

$$c_4 = \tfrac{1}{39813120}(5\alpha^8 - 1892\alpha^6 - 239154\alpha^4 + 2723452\alpha^2 - 2482411), \ \ldots\ .$$

We observe that when $\alpha = 1$, we have $\psi(x) = e^x$ and the coefficients c_s with $s \ge 1$ correctly vanish. Although the analysis in Section 4 applies only to integer values of α, it can be shown that (5.1) holds for arbitrary finite $\alpha > 0$. Consequently, the expansion for $\psi(x)$ in (5.2) holds for $\alpha > 0$.

Example 2. The second example concerns the behaviour of the integral

$$I_m = \int_{-\infty}^\infty e^{-x^2} H_m^4(x)\, dx \qquad (m \to \infty),$$

where $H_m(x)$ denotes the Hermite polynomial of order m. This integral arose in a combinatoric problem studied in [1]. These authors derived an asymptotic estimate by expressing I_m as an integral involving a Legendre function taken round a contour surrounding the origin in the complex plane, and from this constructed a generating function to which they applied Darboux's method. The asymptotics of I_m as $m \to \infty$ were then deduced from the bahaviour of the generating function at its singularities on its circle of convergence.

Here, we shall obtain an asymptotic expansion for I_m by means of the discrete analogue of Laplace's method. It is shown in [1, 10] that

$$I_m = 2^{2m}(m!)^4 \sum_{n=0}^m \frac{2^{2n}\Gamma(n+\frac{1}{2})}{(n!)^3((m-n)!)^2}. \tag{5.3}$$

This sum consists of positive terms which are easily shown to possess a maximum for large m at $n \simeq \frac{2}{3}m$. For arbitrary $0 < \epsilon < \frac{1}{2}$ and $N = \lceil \frac{2}{3}m \rceil$, we then have

$$S_m := \sum_{n=0}^m \frac{2^{2n}\Gamma(n+\frac{1}{2})}{(n!)^3((m-n)!)^2} \sim \sum_{n=[(1-\epsilon)N]}^{[(1+\epsilon)N]} \frac{2^{2n}\Gamma(n+\frac{1}{2})}{(n!)^3((m-n)!)^2} \tag{5.4}$$

with an error that is subdominant with respect to every power of $1/m$ as $m \to \infty$ (we omit these details).

From (4.2) and (4.3) with the parameter $a = \frac{1}{2}$, we find for $n \to \infty$

$$\frac{2^{2n}\Gamma(n+\frac{1}{2})}{(n!)^3} \sim \frac{2^{2n}n^{-2n}e^{2n}}{2\pi n^{3/2}} \left(1 - \frac{7}{24n} + \frac{49}{1152n^2} + \cdots\right).$$

We now let $\nu \equiv \frac{2}{3}m = \lceil \nu \rceil - \delta$, where[3] $0 \leq \delta < 1$, and define the variables

$$n = \nu + t, \quad t = k + \delta, \quad u = t/\nu, \quad \tau_k = (k+\delta)/\nu^{1/2},$$

where $|t|$ is small compared with ν. Since $m - n = \frac{1}{2}\nu - t = \frac{1}{2}\nu(1-2u)$, we have from (4.2)

$$(m-n)! \sim \sqrt{2\pi}(\tfrac{1}{2}\nu - t)^{\frac{1}{2}\nu - t + \frac{1}{2}} e^{-\frac{1}{2}\nu + t} \sum_{s=0}^{\infty} \frac{(-)^s 2^s \gamma_s}{\nu^s (1-2u)^s}$$

for large ν. Some routine but laborious algebra then shows that the terms in the second sum in (5.4) in the neighbourhood of the maximum term can be expanded as

$$\frac{2^{2n}\Gamma(n+\frac{1}{2})}{(n!)^3((m-n)!)^2} \sim \frac{3}{4\pi^2 m}\left(\frac{3}{2m}\right)^{3/2}\left(\frac{m}{3}\right)^{-2m} e^{2m-3\nu u^2}$$

$$\times \frac{(1+u)^{-3/2}}{1-2u} \, \frac{\exp\left[3\nu u^2 - 2\nu(1+u)\log(1+u)\right]}{\exp\left[\nu(1-2u)\log(1-2u)\right]} \, G(u,\nu),$$

where

$$G(u,\nu) = \frac{\left(1 - \frac{7(1+u)^{-1}}{24\nu} + \frac{49(1+u)^{-2}}{1152\nu^2} + \cdots\right)}{\left(1 + \frac{(1-2u)^{-1}}{6\nu} + \frac{(1-2u)^{-2}}{72\nu^2} + \cdots\right)^2}.$$

This finally produces the expansion for $m \to \infty$ in the form

$$\frac{2^{2n}\Gamma(n+\frac{1}{2})}{(n!)^3((m-n)!)^2} \sim \frac{3}{4\pi^2 m}\left(\frac{3}{2m}\right)^{3/2}\left(\frac{m}{3}\right)^{-2m} e^{2m-3\tau_k^2} \sum_{s=0}^{\infty} \frac{P_s(\tau_k)}{\nu^{s/2}}, \tag{5.5}$$

where, omitting the odd-order coefficients,

$$P_0(\tau_k) = 1, \qquad P_2(\tau_k) = -\tfrac{5}{8} + \tfrac{23}{8}\tau_k^2 - 2\tau_k^4 + \tfrac{1}{2}\tau_k^6,$$

$$P_4(\tau_k) = \tfrac{25}{128} - \tfrac{231}{64}\tau_k^2 + \tfrac{1435}{128}\tau_k^4 - \tfrac{891}{80}\tau_k^6 + \tfrac{77}{16}\tau_k^8 - \tfrac{5}{6}\tau_k^{10} + \tfrac{1}{24}\tau_k^{12}, \; \ldots \; .$$

We now extend the range of summation in (5.4) to $\pm\infty$ (with the introduction of exponentially small errors by Lemma 1) to obtain

$$S_m \sim \frac{3e^{2m}}{4\pi^2 m}\left(\frac{3}{2m}\right)^{3/2}\left(\frac{m}{3}\right)^{-2m} \sum_{k=-\infty}^{\infty} e^{-3\tau_k^2}\left(1 + \frac{P_1(\tau_k)}{\nu^{1/2}} + \frac{P_2(\tau_k)}{\nu} + \cdots\right).$$

The sums appearing in this last expression may be evaluated by Lemma 2. We have from (2.5)

$$\sum_{k=-\infty}^{\infty} \tau_k^{2r} e^{-3\tau_k^2} = \sum_{k=-\infty}^{\infty} \frac{(k+\delta)^{2r}}{\nu^r} e^{-3(k+\delta)^2/\nu} \sim (2\pi m)^{1/2}\, 3^{-r-1} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

as $\nu \to \infty$, from which we may derive the expansion

$$S_m \sim \frac{3e^{2m}}{4\pi^2 m}\left(\frac{3}{2m}\right)^{3/2}\left(\frac{m}{3}\right)^{-2m} \sum_{s=0}^{\infty} b_s m^{-s},$$

[3]The value of $\delta = 0$, $\frac{1}{3}$ or $\frac{2}{3}$ according to the value of m.

where $b_0 = 1$, $b_1 = -\frac{5}{12}$, $b_2 = \frac{17}{144}$. Continuation of the expansion process in (5.5) with the help of *Mathematica* then produces the coefficients

$$b_3 = \tfrac{287}{12960}, \quad b_4 = \tfrac{1301}{62208}, \quad b_5 = \tfrac{1371821}{26127360}, \; \ldots .$$

From (5.3), we then obtain

$$I_m \sim \frac{3}{4m}\sqrt{\frac{3}{\pi}}\, 6^{2m}(m!)^2 \left(\sum_{s=0}^{\infty} b_s m^{-s}\right)\left(\sum_{s=0}^{\infty} (-)^s \gamma_s m^{-s}\right)^2,$$

where we have replaced a factor of $(m!)^2$ with the help of (4.2). This finally yields the expansion[4]

$$I_m \sim \frac{3}{4m}\sqrt{\frac{3}{\pi}}\, 6^{2m}(m!)^2 \sum_{s=0}^{\infty} c_s m^{-s} \qquad (m \to \infty), \tag{5.6}$$

where

$$c_0 = 1, \quad c_1 = -\tfrac{1}{4}, \quad c_2 = \tfrac{1}{16}, \quad c_3 = \tfrac{1}{32},$$
$$c_4 = \tfrac{7}{256}, \quad c_5 = \tfrac{59}{1024}, \; \ldots .$$

The first three terms of this expansion were obtained in [1], where the third coefficient was incorrectly given as $c_2 = \frac{3}{16}$. To demonstrate the validity of this expansion we present in Table 1 the absolute relative error in the computation of I_m in (5.3) by means of (5.6) for different values of m and truncation index s.

s	$m = 50$	$m = 100$	$m = 200$
0	5.000×10^{-3}	2.500×10^{-3}	1.250×10^{-3}
1	2.538×10^{-5}	6.297×10^{-6}	1.568×10^{-6}
2	2.558×10^{-7}	3.161×10^{-8}	3.928×10^{-9}
3	4.592×10^{-9}	2.801×10^{-10}	1.729×10^{-11}
4	1.956×10^{-10}	5.931×10^{-12}	1.827×10^{-13}
5	1.026×10^{-11}	1.552×10^{-13}	2.387×10^{-15}

Table 1: Values of the absolute relative error in the computation of I_m by the asymptotic expansion (5.6) for different values of m and truncation index s.

6. Concluding remarks

In Sections 3 and 4 we gave a justification of the use of the discrete analogue of Laplace's method applied to the series of positive terms $F(x)$ in (1.1). The evaluation of the dominant contribution to $F(x)$, which results from the terms in the series near the greatest term, was carried out by means of the Poisson-Jacobi transformation, thereby avoiding the complications inherent in expressing the sum as an integral with limits extended to $\pm\infty$, as employed by Stokes [12] and in [2, p. 304].

We remark that the same procedure can be applied to the more general function, known as the Wright function, defined by

$${}_p\Psi_q(x) = \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} \Gamma(\alpha_r n + a_r)}{\prod_{r=1}^{q} \Gamma(\beta_r n + b_r)} \frac{x^n}{n!}, \tag{5.1}$$

[4] An obvious misprint in [1, Eq. (58)] has the factor $(m!)^2$ in the denominator.

where p, q are nonnegative integers and the parameters α_r, β_r, a_r and b_r are (here) real and positive. Similar arguments then lead to the exponential expansion

$$ {}_p\Psi_q(x) \sim A_0 X^{\vartheta} e^X \sum_{s=0}^{\infty} c_s X^{-s}, \qquad X = \kappa(hx)^{1/\kappa} $$

as $x \to +\infty$, where

$$ \kappa = 1 + \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{p} \alpha_r, \quad h = \prod_{r=1}^{p} \alpha_r^{\alpha_r} \prod_{r=1}^{q} \beta_r^{-\beta_r}, \quad \vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \tfrac{1}{2}(q-p), $$

$$ A_0 = (2\pi)^{\frac{1}{2}(p-q)} \kappa^{-\frac{1}{2}-\vartheta} \prod_{r=1}^{p} \alpha_r^{a_r-\frac{1}{2}} \prod_{r=1}^{q} \beta_r^{\frac{1}{2}-b_r}. $$

The first two coefficients c_s are given by

$$ c_0 = 1, \qquad c_1 = \tfrac{1}{2}\kappa(\mathcal{A} + \tfrac{1}{6}\mathcal{B}), $$

where

$$ \mathcal{A} = \sum_{r=1}^{p} \frac{a_r(a_r-1)}{\alpha_r} - \sum_{r=1}^{q} \frac{b_r(b_r-1)}{\beta_r} - \frac{\vartheta}{\kappa}(\vartheta - 1), \qquad \mathcal{B} = \sum_{r=1}^{p} \alpha_r^{-1} - \sum_{r=1}^{q} \beta_r^{-1} + \kappa^{-1} - 1, $$

as found in [9, Appendix A] by a different argument. It is seen that when $\alpha_r = \beta_r = 1$, the coefficient c_1 agrees with that in (4.15) when due account is made for the additional factor $n!$ in the denominator of (5.1).

Acknowledgement The author wishes to acknowledge A. D. Wood for having brought this problem to his attention.

References

[1] R. Azor, J. Gillis and J. D. Victor, Combinatorial applications of Hermite polynomials, SIAM J. Math. Anal. **13** (1982) 879–890.

[2] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.

[3] B. L. J. Braaksma, Asymptotic expansions and analytic continuations for a class of Barnes-integrals, Compos. Math. **15** (1963) 239–341.

[4] E. T. Copson, Asymptotic Expansions, Cambridge University Press, Cambreidge 1965.

[5] G. H. Hardy, On the zeroes of a class of integral functions, Messenger of Math. **34** (1905) 97–101.

[6] W. K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, Canad. Bull. Math. **17** (1974) 317–358.

[7] Y. L. Luke, Special Functions and Their Approximations, Vol. 1, Academic Press, New York 1969.

[8] F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974. Reprinted in A. K. Peters, Massachussets, 1997.

[9] R. B. Paris, Exponentially small expansions in the asymptotics of the Wright function, J. Comput. Appl. Math. **234** (2010) 488–504.

[10] R. B. Paris, Asymptotics of integrals of Hermite polynomials, Appl. Math. Sci. **4** (2010) 3043–3056.

[11] R. B. Paris and D. Kaminski, Asymptotics and Mellin-Barnes Integrals, Cambridge University Press, Cambridge, 2001.

[12] G. G. Stokes, Note on the determination of arbitrary constants which appear as multipliers of semi-convergent series, Proc. Camb. Phil. Soc. **6** (1889) 362–366.

[13] X. S. Wang and R. Wong, Discrete analogues of Laplace's approximation, Asymptotic Anal. **54** (2007) 165–180.

[14] G. N. Watson, A Treatise on Bessel Functions, Cambridge University Press, Cambridge 1952.

[15] E. T. Whittaker and G. N. Watson, Modern Analysis, Cambridge University Press, Cambridge 1952.

[16] R. Wong, Asymptotic Approximation of Integrals, Academic Press, Boston 1989.

[17] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, Proc. Lond. Math. Soc. (Ser. 2) **10** (1935) 286–293.