Clausen's series ${}_3F_2(1)$ with integral parameter differences and transformations of the hypergeometric function ${}_2F_2(x)$

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Abstract

We obtain summation formulas for the hypergeometric series ${}_3F_2(1)$ with at least one pair of numeratorial and denominatorial parameters differing by a negative integer. The results derived for the latter are used to obtain Kummer-type transformations for the generalized hypergeometric function ${}_2F_2(x)$ and reduction formulas for certain Kampé de Fériet functions. Certain summations for the partial sums of the Gauss hypergeometric series ${}_2F_1(1)$ are also obtained.

Mathematics Subject Classification: 33C15, 33C20, 33C50

Keywords: Generalized hypergeometric function, summation and reduction formula, Kummer-type transformation, partial sums of the Gauss function at unit argument

1. Introduction

In this paper we consider Clausen's hypergeometric function ${}_3F_2(x)$ evaluated at unit argument $x = 1$, namely

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ f, g \end{matrix} \middle| 1\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(f)_k (g)_k \, k!}, \tag{1.1} $$

where at least one pair of numeratorial and denominatorial parameters differs by a negative integer. The Pochhammer symbol, or ascending factorial, $(a)_k$ is defined for all integers k (positive, negative and zero) and complex a by

$$ (a)_k \equiv \frac{\Gamma(a+k)}{\Gamma(a)}, $$

where $\Gamma(x)$ is the gamma function. The series ${}_3F_2(1)$ defined by (1.1) always converges provided that either the parametric excess $s \equiv f + g - a - b - c$ is such that $\mathrm{Re}(s) > 0$ or one of the numeratorial parameters is a negative integer.

When a pair of numeratorial and denominatorial parameters differs by a positive integer m, Karlsson [3] has deduced the summation formula

$$ {}_3F_2\left(\begin{matrix} a, b, & f+m \\ c, & f \end{matrix} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k (a)_k (b)_k}{(f)_k (1+a+b-c)_k}, \tag{1.2} $$

where here and in the sequel we assume that the parameters of the hypergeometric series ${}_3F_2(1)$ are such that it converges and the summation formula for it makes sense. A generalization of (1.2) has been obtained in [9] and rederived in [10] in a somewhat simpler form, namely

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \; (f_r+m_r) \\ c, \quad (f_r) \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \alpha_k \frac{(-1)^k (a)_k (b)_k}{(1+a+b-c)_k}, \tag{1.3} $$

where $m \equiv m_1 + \cdots + m_r$, (f_r) denotes the parameter sequence $(f_1, \ldots, f_r)$ and it can be shown [11] for $0 \le k \le m$ that

$$ \alpha_k = \frac{(-1)^k}{k!} \, {}_{r+1}F_r\left(\begin{matrix} -k, \; (f_r+m_r) \\ (f_r) \end{matrix} \middle| 1 \right). \tag{1.4} $$

In view of the summation formula (1.2) one may ask for an analogous formula for ${}_3F_2(1)$ where at least one pair of numeratorial and denominatorial parameters differs by a negative integer. To this end we shall derive in the present investigation the following summation formulas for positive integers n and p, namely

$$ {}_3F_2\left(\begin{matrix} a, b, \quad n \\ c, \quad n+1 \end{matrix} \middle| 1 \right) = \frac{n!(1-c)_n}{(1-a)_n(1-b)_n} - \frac{\Gamma(c)\Gamma(1+c-a-b)}{\Gamma(c-a)\Gamma(1+c-b)} \sum_{k=1}^{n} \frac{(-n)_k (b-c)_k}{(b-n)_k (1-a)_k}, \tag{1.5} $$

$$ {}_3F_2\left(\begin{matrix} a, b, \quad c \\ b+n, \; c+1 \end{matrix} \middle| 1 \right) = \frac{c\Gamma(1-a)(b)_n}{(b-c)_n} \left(\frac{\Gamma(c)}{\Gamma(1+c-a)} - \frac{\Gamma(b)}{\Gamma(1+b-a)} \sum_{k=0}^{n-1} \frac{(1-a)_k (b-c)_k}{(1+b-a)_k \, k!} \right) \tag{1.6} $$

and

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a, b, \quad n \\ c, \quad n+p \end{matrix} \middle| 1 \right) &= (p)_n \sum_{k=0}^{p-1} (-1)^k (n)_k \binom{p-1}{k} \frac{(1-c)_{k+n}}{(1-a)_{k+n}(1-b)_{k+n}} \\ &\quad + (n)_p \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{n-1} (-1)^k (p)_k \binom{n-1}{k} \frac{(c-a-b)_{k+p}}{(1-a)_{k+p}(1-b)_{k+p}}. \end{aligned} \tag{1.7} $$

Equations (1.5) [6, (B.5)] and (1.6) [5, (11)] are conjectures due to Milgram who deduced them by means of experimental mathematics. We shall prove (1.5), (1.6) and an equivalent form of (1.6) in Section 3. The summation formula (1.7) will be proved in Section 4. Finally, in Section 5 we shall employ these summation theorems to deduce four Kummer-type transformations for the hypergeometric function ${}_2F_2(x)$ and certain reduction formulas for the Kampé de Fériet function. In the next Section 2 we shall establish four preliminary lemmas and a corollary which is a consequence of (1.5)

2. Preliminary results

We shall first prove the specialization $p = 1$ of (1.7).

Lemma 1. *For positive integers n we have the summation formula*

$$ {}_3F_2\left(\begin{matrix} a, b, \quad n \\ c, \quad n+1 \end{matrix} \middle| 1 \right) = \frac{n!(1-c)_n}{(1-a)_n(1-b)_n} - \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=1}^{n} \frac{(-n)_k (c-a-b)_k}{(1-a)_k (1-b)_k}, \tag{2.1} $$

provided $Re(c-a-b) > -1$.

Proof: Consider

$${}_3F_2\left(\begin{matrix} a, b, & f \\ c, & f+1 \end{matrix} \middle| x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{(f)_k}{(f+1)_k} \frac{x^k}{k!}.$$

Since $(f)_k/(f+1)_k = f/(f+k)$, we have

$$x^f \, {}_3F_2\left(\begin{matrix} a, b, & f \\ c, & f+1 \end{matrix} \middle| x\right) = f \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (f+k)} \frac{x^{f+k}}{k!}.$$

Then differentiation with respect to x of both sides of this equation and division of the result by fx^{f-1} gives

$$\frac{abx}{c(f+1)} \, {}_3F_2\left(\begin{matrix} a+1, b+1, & f+1 \\ c+1, & f+2 \end{matrix} \middle| x\right) + {}_3F_2\left(\begin{matrix} a, b, & f \\ c, & f+1 \end{matrix} \middle| x\right) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right).$$

Now setting $x = 1$ and assuming $\mathrm{Re}(c - a - b) > -1$, we have the recurrence relation

$${}_3F_2\left(\begin{matrix} a+1, b+1, & f+1 \\ c+1, & f+2 \end{matrix} \middle| 1\right) = \frac{c(f+1)}{ab}\left[{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1\right) - {}_3F_2\left(\begin{matrix} a, b, & f \\ c, & f+1 \end{matrix} \middle| 1\right)\right]. \tag{2.2}$$

We shall prove (2.1) by induction. For $n = 1$, (2.1) reduces to

$${}_3F_2\left(\begin{matrix} a, b, 1 \\ c, \ \ 2 \end{matrix} \middle| 1\right) = \frac{(1-c)}{(1-a)(1-b)}\left(1 - \frac{\Gamma(c-1)\Gamma(1+c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)$$

which is true and may be obtained by specializing (2.5) with $p = 2$, $q = 1$, $x = 1$ and an application of the Gauss summation theorem for ${}_2F_1(1)$; see [4, 3.13.3 (41)]. Suppose (2.1) is true for an arbitrary but fixed positive integer n. This is the induction hypothesis. Then by the recurrence relation (2.2) and the induction hypothesis we have for $f = n$

$$\begin{aligned}
{}_3F_2\left(\begin{matrix} a, b, & n+1 \\ c, & n+2 \end{matrix} \middle| 1\right) &= \frac{(c-1)(n+1)}{(a-1)(b-1)}\left[{}_2F_1\left(\begin{matrix} a-1, b-1 \\ c-1 \end{matrix} \middle| 1\right) - {}_3F_2\left(\begin{matrix} a-1, b-1, & n \\ c-1, & n+1 \end{matrix} \middle| 1\right)\right] \\
&= \frac{(c-1)(n+1)}{(a-1)(b-1)}\left(\frac{\Gamma(c-1)\Gamma(1+c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - \frac{n!(2-c)_n}{(2-a)_n(2-b)_n}\right. \\
&\qquad \left. + \frac{\Gamma(c-1)\Gamma(1+c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=1}^{n} \frac{(-n)_k (1+c-a-b)_k}{(2-a)_k (2-b)_k}\right),
\end{aligned}$$

where we have employed the Gauss summation formula to evaluate ${}_2F_1(1)$. However

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(-n)_k (1+c-a-b)_k}{(2-a)_k (2-b)_k} &= -\frac{(1-a)(1-b)}{(n+1)(c-a-b)} \sum_{k=2}^{n+1} \frac{(-n-1)_k (c-a-b)_k}{(1-a)_k (1-b)_k} \\
&= -\frac{(1-a)(1-b)}{(n+1)(c-a-b)}\left(\sum_{k=1}^{n+1} \frac{(-n-1)_k (c-a-b)_k}{(1-a)_k (1-b)_k} + \frac{(n+1)(c-a-b)}{(1-a)(1-b)}\right),
\end{aligned}$$

which when used together with the previous result yields upon simplification

$${}_3F_2\left(\begin{matrix} a, b, & n+1 \\ c, & n+2 \end{matrix} \middle| 1\right) = \frac{(n+1)!(1-c)_{n+1}}{(1-a)_{n+1}(1-b)_{n+1}} - \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=1}^{n+1} \frac{(-n-1)_k (c-a-b)_k}{(1-a)_k (1-b)_k}.$$

Thus by the principle of mathematical induction this completes the proof. □

Following Slater [14, p. 83] we define for nonnegative integers n the partial sum to $n+1$ terms of the Gauss function of unit argument

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1\right)_n \equiv \sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k k!}$$

and prove the following.

Lemma 2. *For nonnegative integers n*

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| 1\right)_n = \frac{(1+b)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n, b, c-a \\ 1+b, c \end{matrix}\middle| 1\right). $$

Proof: From [14, (2.6.3), p. 81] and [1, Corollary 3.3.5, p. 142] we have respectively

$$ \sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k k!} = \frac{\Gamma(1+a+n)\Gamma(1+b+n)}{\Gamma(n+1)\Gamma(1+a+b+n)}\, {}_3F_2\left(\begin{matrix} a, b, c+n \\ 1+a+b+n, c \end{matrix}\middle| 1\right) $$

and

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ f, g \end{matrix}\middle| 1\right) = \frac{\Gamma(g)\Gamma(s)}{\Gamma(g-a)\Gamma(s+a)}\, {}_3F_2\left(\begin{matrix} a, f-b, f-c \\ f, s+a \end{matrix}\middle| 1\right), $$

where s is the parametric excess. If in this last identity we replace $c \mapsto c+n$ and then let $f = c$, $g = 1+a+b+n$, we find

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a, b, c+n \\ 1+a+b+n, c \end{matrix}\middle| 1\right) &= \frac{\Gamma(1+a+b+n)\Gamma(1)}{\Gamma(1+b+n)\Gamma(1+a)}\, {}_3F_2\left(\begin{matrix} a, c-b, -n \\ c, 1+a \end{matrix}\middle| 1\right) \\ &= \frac{\Gamma(1+a+b+n)\Gamma(1)}{\Gamma(1+a+n)\Gamma(1+b)}\, {}_3F_2\left(\begin{matrix} b, c-a, -n \\ c, 1+b \end{matrix}\middle| 1\right) \end{aligned} $$

upon interchanging the parameters a and b.

Hence

$$ \sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k k!} = \frac{\Gamma(1+b+n)}{\Gamma(1+b)\Gamma(n+1)}\, {}_3F_2\left(\begin{matrix} -n, b, c-a \\ 1+b, c \end{matrix}\middle| 1\right) $$

and this proves the lemma. □

Lemma 3. *For positive integers n*

$$ \frac{c-a-b}{c-b} \sum_{k=1}^{n} \frac{(-n)_k (b-c)_k}{(b-n)_k (1-a)_k} = \sum_{k=1}^{n} \frac{(-n)_k (c-a-b)_k}{(1-a)_k (1-b)_k}. \tag{2.3} $$

Proof: It is evident by adjusting the summation index that

$$ \sum_{k=1}^{n} \frac{(-n)_k (b-c)_k}{(b-n)_k (1-a)_k} = \frac{n(c-b)}{(b-n)(1-a)}\, {}_3F_2\left(\begin{matrix} 1-n, 1, 1+b-c \\ 1+b-n, 2-a \end{matrix}\middle| 1\right) $$

and

$$ \sum_{k=1}^{n} \frac{(-n)_k (c-a-b)_k}{(1-a)_k (1-b)_k} = -\frac{n(c-a-b)}{(1-a)(1-b)}\, {}_3F_2\left(\begin{matrix} 1-n, 1, 1+c-a-b \\ 2-a, 2-b \end{matrix}\middle| 1\right). $$

Thus, upon replacing $n \mapsto n+1$, we see that (2.3) is equivalent to

$$ {}_3F_2\left(\begin{matrix} -n, 1, 1+b-c \\ 2-a, b-n \end{matrix}\middle| 1\right) = \frac{n+1-b}{1-b}\, {}_3F_2\left(\begin{matrix} -n, 1, 1+c-a-b \\ 2-a, 2-b \end{matrix}\middle| 1\right). \tag{2.4} $$

This result may be obtained from the transformation [1, p. 142]

$$ {}_3F_2\left(\begin{matrix} -n, \alpha, \beta \\ \delta, \epsilon \end{matrix}\middle| 1\right) = \frac{(\epsilon-\alpha)_n}{(\epsilon)_n}\, {}_3F_2\left(\begin{matrix} -n, \alpha, \delta-\beta \\ \delta, 1+\alpha-\epsilon-n \end{matrix}\middle| 1\right) $$

with $\alpha = 1$, $\beta = 1+b-c$, $\delta = 2-a$ and $\epsilon = b-n$. Thus (2.3) and (2.4) are equivalent and this proves the lemma. □

We define the product of p Pochhammer symbols

$$ ((a_p))_k \equiv (a_1)_k \cdots (a_p)_k, $$

where when $p = 0$ the product reduces to unity.

Lemma 4. *For positive integers n*

$$ {}_{p+1}F_{q+1}\left(\begin{matrix}(a_p), & 1\\ (b_q), & n+1\end{matrix}\,\middle|\, x\right) = (-1)^{n(p-q)}\frac{n!}{x^n}\,\frac{((1-b_q))_n}{((1-a_p))_n} $$
$$ \times\left[{}_pF_q\left(\begin{matrix}(a_p-n)\\ (b_q-n)\end{matrix}\,\middle|\, x\right) - \sum_{k=0}^{n-1}\frac{((a_p-n))_k}{((b_q-n))_k}\,\frac{x^k}{k!}\right]. \tag{2.5} $$

Proof: Let us denote the left-hand side of (2.5) by S. Then, since

$$ \frac{1}{(n+1)_k} = \frac{n!}{(1)_{n+k}}, $$

we have

$$ S = n!\sum_{k=0}^{\infty}\frac{((a_p))_k}{((b_q))_k}\,\frac{x^k}{(1)_{n+k}} = n!\sum_{k=n}^{\infty}\frac{((a_p))_{k-n}}{((b_q))_{k-n}}\,\frac{x^{k-n}}{k!}. $$

Use of the identity

$$ (\alpha)_{k-n} = \frac{(-1)^n(\alpha-n)_k}{(1-\alpha)_n} $$

then yields

$$ S = (-1)^{n(p-q)}\frac{n!}{x^n}\,\frac{((1-b_q))_n}{((1-a_p))_n}\sum_{k=n}^{\infty}\frac{((a_p-n))_k}{((b_q-n))_k}\,\frac{x^k}{k!} $$

and the lemma follows. □

We remark that (2.5) is recorded incorrectly in [13, 7.2.3 (19)], but certainly the result is well known. Although we shall prove (1.5) in Section 3 we now exploit the $a \leftrightarrow b$ symmetry of this result to obtain the following.

Corollary 1. *For nonnegative integers n*

$$ {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\,\middle|\, 1\right)_n = \frac{(1+a)_n(1+b)_n}{(c)_n n!}\sum_{k=0}^{n}\frac{(-n)_k(1+a+b-c)_k}{(1+a)_k(1+b)_k}. \tag{2.6} $$

Proof: Since the right-hand side of (1.5) must remain the same when the parameters a and b are interchanged, it follows that

$$ \frac{1}{c-b}\sum_{k=1}^{n}\frac{(-n)_k(b-c)_k}{(b-n)_k(1-a)_k} = \frac{1}{c-a}\sum_{k=1}^{n}\frac{(-n)_k(a-c)_k}{(a-n)_k(1-b)_k}. $$

Now replace $a \mapsto 1-a+n$ and $c \mapsto 1+b-c+n$ so that

$$ \frac{1}{c-1-n}\sum_{k=1}^{n}\frac{(-n)_k(c-1-n)_k}{(b-n)_k(a-n)_k} = \frac{1}{c-a-b}\sum_{k=1}^{n}\frac{(-n)_k(c-a-b)_k}{(1-a)_k(1-b)_k}. $$

By reversing the index $k \mapsto n-k$ in the left-hand summation and employing the identity

$$ (\alpha-n)_{n-k} = (-1)^{n-k}\frac{(1-\alpha)_n}{(1-\alpha)_k}, $$

we have upon replacing $a \mapsto 1-a$, $b \mapsto 1-b$ and $c \mapsto 2-c$

$$ \sum_{k=0}^{n-1}\frac{(a)_k(b)_k}{(c)_k k!} = \frac{1}{c-a-b}\,\frac{(a)_n(b)_n}{(c)_{n-1}\,n!}\sum_{k=1}^{n}\frac{(-n)_k(a+b-c)_k}{(a)_k(b)_k}. $$

Finally, by adjusting the index k in the right-hand summation and then letting $n \mapsto n+1$ in the resulting equation, we deduce (2.6). □

3. Proof of Milgram's conjectures

That Milgram's (1.5) is equivalent to (2.1) is seen by means of Lemma 3. Moreover, Milgram suggests that (1.5) can be obtained by employing the Thomae two-term relation [1, Corollary 3.3.6, p. 143]

$$ {}_3F_2\left(\begin{matrix} a,b,c \\ f,g \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(f)\Gamma(g)\Gamma(s)}{\Gamma(b)\Gamma(s+a)\Gamma(s+c)}\, {}_3F_2\left(\begin{matrix} s, f-b, g-b \\ s+a, s+c \end{matrix}\,\middle|\, 1\right) $$

together with the conjecture (1.6) and this does indeed yield (1.5). Next we shall give a derivation of (1.6), which we restate below.

Theorem 1. *For positive integers n*

$$ {}_3F_2\left(\begin{matrix} a, b, & c \\ b+n, & c+1 \end{matrix}\,\middle|\, 1\right) = \frac{c\Gamma(1-a)(b)_n}{(b-c)_n}\left(\frac{\Gamma(c)}{\Gamma(1+c-a)} - \frac{\Gamma(b)}{\Gamma(1+b-a)}\sum_{k=0}^{n-1}\frac{(1-a)_k(b-c)_k}{(1+b-a)_k\, k!}\right). $$

Proof: We employ one of Thomae's three-term relations for ${}_3F_2(1)$ given by [13, 7.4.4 (3)]

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a,b,c \\ f,g \end{matrix}\,\middle|\, 1\right) &= \frac{\Gamma(f)\Gamma(g)\Gamma(c-b)}{\Gamma(c)\Gamma(f-b)\Gamma(g-b)}\,\frac{\Gamma(1-a)}{\Gamma(1+b-a)}\, {}_3F_2\left(\begin{matrix} b, 1+b-f, 1+b-g \\ 1+b-a, 1+b-c \end{matrix}\,\middle|\, 1\right) \\ &+ \frac{\Gamma(f)\Gamma(g)\Gamma(b-c)}{\Gamma(b)\Gamma(f-c)\Gamma(g-c)}\,\frac{\Gamma(1-a)}{\Gamma(1+c-a)}\, {}_3F_2\left(\begin{matrix} c, 1+c-f, 1+c-g \\ 1+c-a, 1+c-b \end{matrix}\,\middle|\, 1\right), \end{aligned} $$

where $\mathrm{Re}(f+g-a-b-c) > 0$. In this set $f = b+n$ and $g = c+1$, thus giving for positive integers n the result

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a, b, & c \\ b+n, & c+1 \end{matrix}\,\middle|\, 1\right) &= \frac{(b)_n}{(b-c)_n}\,\frac{\Gamma(1+c)\Gamma(1-a)}{\Gamma(1+c-a)} \\ &+ \frac{c(b)_n}{(c-b)\Gamma(n)}\,\frac{\Gamma(b)\Gamma(1-a)}{\Gamma(1+b-a)}\, {}_3F_2\left(\begin{matrix} 1-n, b, b-c \\ 1+b-a, 1+b-c \end{matrix}\,\middle|\, 1\right). \end{aligned} \tag{3.1} $$

Now in Lemma 2 let $a \mapsto 1-a$, $b \mapsto b-c$, $c \mapsto 1+b-a$ and $n \mapsto n-1$ thus yielding

$$ \begin{aligned} \sum_{k=0}^{n-1}\frac{(1-a)_k(b-c)_k}{(1+b-a)_k k!} &= \frac{(1+b-c)_{n-1}}{(n-1)!}\, {}_3F_2\left(\begin{matrix} 1-n, b-c, b \\ 1+b-c, 1+b-a \end{matrix}\,\middle|\, 1\right) \\ &= \frac{(b-c)_n}{(b-c)\Gamma(n)}\, {}_3F_2\left(\begin{matrix} 1-n, b, b-c \\ 1+b-a, 1+b-c \end{matrix}\,\middle|\, 1\right), \end{aligned} $$

which when used together with (3.1) completes the proof of the theorem. □

It appears that Theorem 1 could also be obtained from [13, 7.2.3 (21)] by specialization, but the latter result is incorrect and, moreover, its origin is unknown.

4. Proof of (1.7)

Employing Lemma 4, we have for positive integers p

$$ {}_3F_2\left(\begin{matrix} a, b, & 1 \\ c, & p+1 \end{matrix}\,\middle|\, x\right) = \xi_p\left[x^{-p}\, {}_2F_1\left(\begin{matrix} a-p, b-p \\ c-p \end{matrix}\,\middle|\, x\right) - \sum_{k=0}^{p-1}\frac{(a-p)_k(b-p)_k}{(c-p)_k k!}x^{k-p}\right], $$

where

$$ \xi_p \equiv \frac{(-1)^p p!(1-c)_p}{(1-a)_p(1-b)_p}. $$

Now differentiation of this identity n times with respect to x yields

$$\frac{(a)_n(b)_n(1)_n}{(c)_n(p+1)_n}\,{}_3F_2\left(\begin{matrix} a+n, b+n, n+1 \\ c+n, n+p+1 \end{matrix}\,\middle|\, x\right) = \xi_p\Big(S_1(x) - S_2(x)\Big),$$

where

$$S_1(x) \equiv \frac{d^n}{dx^n}\left[x^{-p}\,{}_2F_1\left(\begin{matrix} a-p, b-p \\ c-p \end{matrix}\,\middle|\, x\right)\right]$$

and

$$S_2(x) \equiv \sum_{k=0}^{p-1} \frac{(a-p)_k(b-p)_k}{(c-p)_k k!}\,\frac{d^n x^{k-p}}{dx^n}.$$

Recalling Leibniz's rule

$$\frac{d^n}{dx^n}u(x)v(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}u(x)}{dx^{n-k}}\,\frac{d^k v(x)}{dx^k}$$

and

$$\frac{d^n}{dx^n}x^\nu = (-1)^n(-\nu)_n x^{\nu-n},$$

we may write

$${}_3F_2\left(\begin{matrix} a+n, b+n, n+1 \\ c+n, n+p+1 \end{matrix}\,\middle|\, x\right) = \frac{(c)_n(p+1)_n\xi_p}{(a)_n(b)_n n!}\Big(S_1(x) - S_2(x)\Big), \tag{4.1}$$

where

$$S_1(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^{k-n-p}(p)_{n-k}\,\frac{(a-p)_k(b-p)_k}{(c-p)_k}\,{}_2F_1\left(\begin{matrix} a-p+k, b-p+k \\ c-p+k \end{matrix}\,\middle|\, x\right), \tag{4.2}$$

$$S_2(x) = \frac{(-1)^n}{x^{n+p}}\sum_{k=0}^{p-1}\frac{(a-p)_k(b-p)_k}{(c-p)_k k!}\,(p-k)_n x^k. \tag{4.3}$$

In (4.1) – (4.3) set $x=1$ and replace $a \mapsto a-n$, $b \mapsto b-n$ and $c \mapsto c-n$. Then by using in (4.2) the Gauss summation theorem

$${}_2F_1\left(\begin{matrix} a-n-p+k, b-n-p+k \\ c-n-p+k \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c-n-p+k)\Gamma(c-a-b+n+p-k)}{\Gamma(c-a)\Gamma(c-b)},$$

we have

$${}_3F_2\left(\begin{matrix} a, b, n+1 \\ c, n+p+1 \end{matrix}\,\middle|\, 1\right) = \frac{(-1)^p(n+p)!}{n!}\,\frac{(c-n)_n}{(a-n)_n(b-n)_n}\,\frac{(1-c+n)_p}{(1-a+n)_p(1-b+n)_p}$$

$$\times\left((-1)^{n+1}S + \frac{\Gamma(c-n-p)}{\Gamma(c-a)\Gamma(c-b)}\,T\right), \tag{4.4}$$

where

$$S \equiv \sum_{k=0}^{p-1}\frac{(a-n-p)_k(b-n-p)_k}{(c-n-p)_k}\,\frac{(p-k)_n}{k!}$$

and

$$T \equiv \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(p)_{n-k}(a-n-p)_k(b-n-p)_k\Gamma(c-a-b+n+p-k).$$

If, in the first sum S in (4.4), we reverse the summation index $k \mapsto p-k$ and note that

$$(a-n-p)_{p-k} = \frac{\Gamma(a)}{\Gamma(a-n-p)} \frac{(-1)^{n+k}}{(1-a)_{n+k}}, \quad (k)_n = \frac{\Gamma(n)}{\Gamma(k)}(n)_k, \quad \frac{1}{(1)_{p-k}} = \frac{(-1)^k(-p)_k}{p!},$$

we obtain

$$S = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \frac{\Gamma(c-n-p)}{\Gamma(a-n-p)\Gamma(b-n-p)} \frac{(-1)^n\Gamma(n)}{p!} \sum_{k=1}^{p} \frac{(1-c)_{n+k}}{(1-a)_{n+k}(1-b)_{n+k}} \frac{(-p)_k(n)_k}{\Gamma(k)}.$$

Now adjusting the summation index so that it starts at $k=0$ and using

$$(-1)^k \frac{(1-p)_k}{k!} = \binom{p-1}{k},$$

we find

$$S = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \frac{\Gamma(c-n-p)}{\Gamma(a-n-p)\Gamma(b-n-p)} \frac{(-1)^{n+1}n!}{(p-1)!}$$
$$\times \sum_{k=0}^{p-1} (-1)^k (n+1)_k \binom{p-1}{k} \frac{(1-c)_{n+k+1}}{(1-a)_{n+k+1}(1-b)_{n+k+1}}. \tag{4.5}$$

Proceeding in a similar manner for the second sum T in (4.4), upon redefining the summation index $k \mapsto n-k$ and using the symmetry of the binomial coefficient, we have

$$T = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(a-n-p)\Gamma(b-n-p)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p)_k \frac{(c-a-b)_{k+p}}{(1-a)_{k+p}(1-b)_{k+p}}. \tag{4.6}$$

Finally, combining (4.4) – (4.6) followed by some routine simplification and then replacing $n \mapsto n-1$, we deduce the summation formula (1.7). □

5. Transformation and reduction formulas

Few Kummer-type transformations (containing a finite number of terms) for the confluent hypergeometric function ${}_2F_2(x)$ are extant in the literature. Miller [7] deduced the two-term Kummer-type transformation due to Exton [2]

$${}_2F_2\left(\begin{matrix} a, & 1+\frac{1}{2}a \\ b, & \frac{1}{2}a \end{matrix} \middle| x\right) = e^x \, {}_2F_2\left(\begin{matrix} b-a-1, & 2+a-b \\ b, & 1+a-b \end{matrix} \middle| -x\right), \tag{5.1}$$

which has been extended by Paris [12] who obtained

$${}_2F_2\left(\begin{matrix} a, & c+1 \\ b, & c \end{matrix} \middle| x\right) = e^x \, {}_2F_2\left(\begin{matrix} b-a-1, & \xi+1 \\ b, & \xi \end{matrix} \middle| -x\right), \tag{5.2}$$

where

$$\xi \equiv \frac{c(1+a-b)}{a-c}.$$

Paris [12] also obtained a multiple-term Kummer-type transformation given by

$${}_2F_2\left(\begin{matrix} a, & c+n \\ b, & c \end{matrix} \middle| x\right) = e^x \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(c)_k} \, {}_2F_2\left(\begin{matrix} b-a, & c+n \\ b, & c+k \end{matrix} \middle| -x\right), \tag{5.3}$$

where n is a positive integer. Moreover, specialization of the result [4, 9.1 (34)] recorded by Luke gives

$$ {}_2F_2\left(\begin{matrix} a, f \\ b, c \end{matrix} \middle| x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (c-f)_k}{(b)_k (c)_k} \frac{(-x)^k}{k!} \, {}_1F_1\left(\begin{matrix} a+k \\ b+k \end{matrix} \middle| x\right), $$

whereupon letting $f = c + n$ and employing Kummer's transformation for the confluent hypergeometric function ${}_1F_1(x)$ we easily obtain for n a positive integer

$$ {}_2F_2\left(\begin{matrix} a, & c+n \\ b, & c \end{matrix} \middle| x\right) = e^x \sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k x^k}{(b)_k (c)_k} \, {}_1F_1\left(\begin{matrix} b-a \\ b+k \end{matrix} \middle| -x\right). \tag{5.4} $$

In each of the results (5.1) – (5.4) one pair of numeratorial and denominatorial parameters in the left-hand ${}_2F_2(x)$ function differs by a positive integer. In what follows we shall employ the summation formulas (1.5) – (1.7) and (3.1) to obtain multiple-term Kummer-type transformations for ${}_2F_2(x)$ in which at least one pair of numeratorial and denominatorial parameters differs by a negative integer. For this, we shall make use of the expansion

$$ {}_2F_2\left(\begin{matrix} a, b \\ c, f \end{matrix} \middle| x\right) = e^x \sum_{k=0}^{\infty} {}_3F_2\left(\begin{matrix} -k, a, b \\ c, f \end{matrix} \middle| 1\right) \frac{(-x)^k}{k!} \tag{5.5} $$

which can be obtained by specialization from a more general result recorded in [16, p. 166]. In (1.5) – (1.7) now let $a = -k$ thus giving for nonnegative integers k respectively

$$ {}_3F_2\left(\begin{matrix} -k, b, & n \\ c, & n+1 \end{matrix} \middle| 1\right) = \frac{(1-c)_n (1)_k}{(1-b)_n (1+n)_k} - \sum_{\ell=1}^{n} \frac{(-n)_\ell (b-c)_\ell}{(b-n)_\ell \, \ell!} \frac{(1+c-b)_k (1)_k}{(c)_k (1+\ell)_k}, \tag{5.6} $$

$$ {}_3F_2\left(\begin{matrix} -k, b, & c \\ b+n, & c+1 \end{matrix} \middle| 1\right) = \frac{(b)_n}{(b-c)_n} \left(\frac{(1)_k}{(1+c)_k} - \frac{c}{b} \sum_{\ell=0}^{n-1} \frac{(b-c)_\ell}{(1+b)_\ell} \frac{(1+\ell)_k}{(1+b+\ell)_k} \right), \tag{5.7} $$

$$ {}_3F_2\left(\begin{matrix} -k, b, & n \\ c, & n+p \end{matrix} \middle| 1\right) = \sum_{\ell=0}^{p-1} A_\ell \frac{(1-c)_{n+\ell}}{(1-b)_{n+\ell}} \frac{(1)_k}{(1+n+\ell)_k} + \sum_{\ell=0}^{n-1} B_\ell \frac{(c-b)_{p+\ell}}{(1-b)_{p+\ell}} \frac{(c-b+p+\ell)_k (1)_k}{(1+p+\ell)_k (c)_k}, \tag{5.8} $$

where for nonegative integers ℓ the coefficients A_ℓ and B_ℓ are defined by

$$ A_\ell \equiv \frac{(p)_n}{(n-1)!} \frac{(1-p)_\ell}{\ell!} \frac{1}{n+\ell}, \qquad B_\ell \equiv \frac{(n)_p}{(p-1)!} \frac{(1-n)_\ell}{\ell!} \frac{1}{p+\ell}. \tag{5.9} $$

Then, employing (5.5) we deduce respectively the multiple-term Kummer-type transformation formulas

$$ {}_2F_2\left(\begin{matrix} b, & n \\ c, & n+1 \end{matrix} \middle| x\right) = e^x \left[\frac{(1-c)_n}{(1-b)_n} \, {}_1F_1\left(\begin{matrix} 1 \\ n+1 \end{matrix} \middle| -x\right) \right. $$

$$ \left. - \sum_{\ell=1}^{n} \frac{(-n)_\ell (b-c)_\ell}{(b-n)_\ell \, \ell!} \, {}_2F_2\left(\begin{matrix} 1+c-b, & 1 \\ c, & 1+\ell \end{matrix} \middle| -x\right) \right], \tag{5.10} $$

$$ {}_2F_2\left(\begin{matrix} b, & c \\ b+n, & c+1 \end{matrix} \middle| x\right) = e^x \frac{(b)_n}{(b-c)_n} \left[{}_1F_1\left(\begin{matrix} 1 \\ c+1 \end{matrix} \middle| -x\right) - \frac{c}{b} \sum_{\ell=0}^{n-1} \frac{(b-c)_\ell}{(1+b)_\ell} \, {}_1F_1\left(\begin{matrix} 1+\ell \\ 1+b+\ell \end{matrix} \middle| -x\right) \right], \tag{5.11} $$

$$ {}_2F_2\left(\begin{matrix} b, & n \\ c, & n+p \end{matrix} \middle| x\right) = e^x \left[\sum_{\ell=0}^{p-1} A_\ell \frac{(1-c)_{n+\ell}}{(1-b)_{n+\ell}} \, {}_1F_1\left(\begin{matrix} 1 \\ 1+n+\ell \end{matrix} \middle| -x\right) \right. $$

$$+\sum_{\ell=0}^{n-1} B_\ell \frac{(c-b)_{p+\ell}}{(1-b)_{p+\ell}}\, {}_2F_2\left(\begin{matrix} c-b+p+\ell, & 1 \\ c, & 1+p+\ell \end{matrix}\,\middle|\, -x\right)\Bigg]. \tag{5.12}$$

In a similar manner we obtain from (3.1) the alternative form of (5.7) given by

$${}_3F_2\left(\begin{matrix} -k, b, & c \\ b+n, & c+1 \end{matrix}\,\middle|\, 1\right) = \frac{(b)_n}{(b-c)_n}\left(\frac{(1)_k}{(1+c)_k} - \frac{c(b-c)_n}{(n-1)!}\right.$$
$$\left.\times \sum_{\ell=0}^{n-1} \frac{(1-n)_\ell}{(b+\ell)(b-c+\ell)\,\ell!}\,\frac{(1)_k}{(1+b+\ell)_k}\right). \tag{5.13}$$

This in turn yields the alternative form of the Kummer-type transformation (5.11), namely

$${}_2F_2\left(\begin{matrix} b, & c \\ b+n, & c+1 \end{matrix}\,\middle|\, x\right) = e^x \frac{(b)_n}{(b-c)_n}\left[{}_1F_1\left(\begin{matrix} 1 \\ c+1 \end{matrix}\,\middle|\, -x\right)\right.$$
$$\left. -\frac{c(b-c)_n}{(n-1)!}\sum_{\ell=0}^{n-1} \frac{(1-n)_\ell}{(b+\ell)(b-c+\ell)\,\ell!}\, {}_1F_1\left(\begin{matrix} 1 \\ 1+b+\ell \end{matrix}\,\middle|\, -x\right)\right]. \tag{5.14}$$

We remark that Prudnikov *et al.* [13, 7.12.1 (2)] records for positive integers n the identity

$${}_2F_2\left(\begin{matrix} b, & c \\ b+n, & c+1 \end{matrix}\,\middle|\, x\right) = \frac{(b)_n}{(b-c)_n}\left[{}_1F_1\left(\begin{matrix} c \\ c+1 \end{matrix}\,\middle|\, x\right) - \frac{c}{b}\sum_{\ell=0}^{n-1} \frac{(b-c)_\ell}{(1+b)_\ell}\, {}_1F_1\left(\begin{matrix} b \\ 1+b+\ell \end{matrix}\,\middle|\, x\right)\right].$$

This also yields, upon use of Kummer's transformation for ${}_1F_1(x)$, the transformation (5.11).

The above transformations for ${}_2F_2(x)$ can also be obtained as specializations of reduction formulas for certain hypergeometric functions in two variables known as the Kampé de Fériet function. The latter function in two complex variables is defined by the double infinite series (see, for example, [15])

$$F^{r:q;u}_{s:t;v}\left(\begin{matrix} (a_r): & (\alpha_q); & (\gamma_u) \\ (b_s): & (\beta_t); & (\delta_v) \end{matrix}\,\middle|\, x, y\right) \equiv \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{((a_r))_{k+\ell}}{((b_s))_{k+\ell}}\,\frac{((\alpha_q))_k}{((\beta_t))_k}\,\frac{((\gamma_u))_\ell}{((\delta_v))_\ell}\,\frac{x^k}{k!}\,\frac{y^\ell}{\ell!}.$$

Thus, employing the identity [8]

$$F^{r:2;0}_{s:2;0}\left(\begin{matrix} (a_r): & a, b\,; & - \\ (b_s): & c, f\,; & - \end{matrix}\,\middle|\, x, -x\right) = \sum_{k=0}^{\infty} \frac{((a_r))_k}{((b_s))_k}\, {}_3F_2\left(\begin{matrix} -k, a, b \\ c, f \end{matrix}\,\middle|\, 1\right) \frac{(-x)^k}{k!} \tag{5.15}$$

together with (5.6) – (5.8) and (5.13) four reduction formulas for the Kampé de Fériet function may be obtained. For brevity, we record only the two salient results obtained respectively from (5.7) and (5.8), namely

$$F^{r:2;0}_{s:2;0}\left(\begin{matrix} (a_r): & b, & c\,; & - \\ (b_s): & b+n, & c+1\,; & - \end{matrix}\,\middle|\, x, -x\right) = \frac{(b)_n}{(b-c)_n}\left[{}_{r+1}F_{s+1}\left(\begin{matrix} (a_r), & 1 \\ (b_s), & c+1 \end{matrix}\,\middle|\, -x\right)\right.$$
$$\left. -\frac{c}{b}\sum_{\ell=0}^{n-1} \frac{(b-c)_\ell}{(1+b)_\ell}\, {}_{r+1}F_{s+1}\left(\begin{matrix} (a_r), & 1+\ell \\ (b_s), & 1+b+\ell \end{matrix}\,\middle|\, -x\right)\right] \tag{5.16}$$

and

$$F^{r:2;0}_{s:2;0}\left(\begin{matrix} (a_r): & b, & n\,; & - \\ (b_s): & c, & n+p\,; & - \end{matrix}\,\middle|\, x, -x\right) = \sum_{\ell=0}^{p-1} A_\ell \frac{(1-c)_{n+\ell}}{(1-b)_{n+\ell}}\, {}_{r+1}F_{s+1}\left(\begin{matrix} (a_r), & 1 \\ (b_s), & 1+n+\ell \end{matrix}\,\middle|\, -x\right)$$
$$+\sum_{\ell=0}^{n-1} B_\ell \frac{(c-b)_{p+\ell}}{(1-b)_{p+\ell}}\, {}_{r+2}F_{s+2}\left(\begin{matrix} (a_r), & c-b+p+\ell, 1 \\ (b_s), & c, 1+p+\ell \end{matrix}\,\middle|\, -x\right), \tag{5.17}$$

where a solid horizontal line indicates an empty parameter sequence, and the coefficients A_ℓ and B_ℓ are given by (5.9).

We remark that (5.5), (5.11) and (5.12) may respectively be obtained from (5.15), (5.16) and (5.17) by specializing the latter three results with $r = s = 0$.

6. Concluding remarks

Although in the present investigation we have derived several conjectured and new summation formulas for the series ${}_3F_2(1)$ in which one pair of numeratorial and denominatorial parameters differs by a negative integer, there remains open the problem of deducing a summation formula for the series

$$ {}_3F_2\left(\begin{matrix} a, b, & f \\ c, & f+n \end{matrix} \middle| 1 \right), $$

where n is a positive integer and a, b, c and f are arbitrary complex numbers. We hope that the developments presented herein will stimulate further interest in this problem.

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