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The Asymptotic Expansion of a Generalised Mathieu Series

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Abstract

We obtain the asymptotic expansion of a generalised Mathieu series and its alternating variant for large complex values of the variable by means of a Mellin transform approach. Numerical examples are presented to demonstrate the accuracy of these expansions.

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1. Introduction

The functional series

$$S_\mu(a) = \sum_{n=1}^{\infty} \frac{n}{(n^2+a^2)^\mu} \qquad (\mu > 1) \tag{1.1}$$

in the case $\mu = 2$ was introduced by Mathieu in his 1890 book [5] dealing with the elasticity of solid bodies. Considerable effort has been devoted to the determination of upper and lower bounds for this series when the parameter $a > 0$; see [10] and the references therein.

Various integral representations have been obtained for the series $S_\mu(a)$, together with its alternating variant

$$\tilde{S}_\mu(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(n^2+a^2)^\mu} \qquad (\mu > \tfrac{1}{2}). \tag{1.2}$$

We have

$$\left.\begin{matrix} S_\mu(a) \\ \tilde{S}_\mu(a) \end{matrix}\right\} = \frac{\pi^{\frac{1}{2}}(2a)^{\frac{3}{2}-\mu}}{2\Gamma(\mu)} \int_0^\infty \frac{t^{\mu-\frac{1}{2}}}{e^t \mp 1} J_{\mu-\frac{3}{2}}(at)\, dt \qquad (a > 0),$$

where $J_\nu(x)$ denotes the Bessel function of the first kind of order ν and the upper or lower sign corresponds to $S_\mu(a)$ or $\tilde{S}_\mu(a)$, respectively. When $\mu = 2$, these reduce to the integrals

$$S_2(a) = \frac{1}{2a} \int_0^\infty \frac{t \sin(at)}{e^t - 1} dt, \qquad \tilde{S}_2(a) = \frac{1}{2a} \int_0^\infty \frac{t \sin(at)}{e^t + 1} dt$$

for $a > 0$, given by Emersleben [2] and Pogány *et al.* [10], respectively. Other integral representations involving the hyperbolic functions have recently been derived in [6].

Our objective in this paper is to derive the asymptotic expansion of the more general functional series[1]

$$\left.\begin{matrix} S_{\mu,\gamma}(a;b) \\ \tilde{S}_{\mu,\gamma}(a;b) \end{matrix}\right\} = \sum_{n=1}^\infty \frac{(\pm 1)^{n-1} n^\gamma}{(n^2 + a^2)^\mu} \exp\left[\frac{-a^2 b}{n^2 + a^2}\right] \qquad (\mu > 0,\ \gamma > -1,\ b \in \mathbf{R}) \tag{1.3}$$

for large complex a, where it is supposed for absolute convergence that $\delta := 2\mu - \gamma > 1$. Since $S_{\mu,\gamma}(a;b)$ and $\tilde{S}_{\mu,\gamma}(a;b)$ are both even in a it is sufficient to consider $|\arg a| \leq \frac{1}{2}\pi$. The series reduce to $S_2(a)$ and $\tilde{S}_2(a)$ when $\gamma = 1$, $b = 0$ and $\mu = 2$. The presence of the exponential $e_n := \exp[-a^2 b/(n^2 + a^2)]$ acts as a perturbing factor which, although it does not affect the rate of convergence of the series (since $e_n \to 1$ as $n \to \infty$), can disturb the extreme cancellation that takes place in the alternating series and so significantly modify its large-a growth. The effect of similar perturbing factors on the negative exponential series for e^{-x} has been discussed in [8], where it was established that the growth for $x \to +\infty$ can be quite different from the unperturbed case. The asymptotic expansions of $S_\mu(a)$ and $\tilde{S}_\mu(a)$ as $a \to +\infty$ have been given in [1, 10]. Asymptotic expansions of $S_{\mu,\gamma}(a;0)$ and $\tilde{S}_{\mu,\gamma}(a;0)$ for $a \to +\infty$ have been considered in [12] using a Mellin transform approach. We adopt a similar approach since it easily enables us to deal with complex a; for a description of this method see, for example, [9, Section 4.1.1].

In the application of the Mellin transform method to the series $S_{\mu,\gamma}(a;b)$ and $\tilde{S}_{\mu,\gamma}(a;b)$ we shall require the following estimates for the gamma function and the Riemann zeta function $\zeta(s)$. For real σ and t, we have the estimates

$$\Gamma(\sigma \pm it) = O(t^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\pi t}), \quad |\zeta(\sigma \pm it)| = O(t^{\omega(\sigma)} \log^\alpha t) \quad (t \to +\infty), \tag{1.4}$$

[1]The restriction $\gamma > -1$ is imposed for convenience to avoid the possible formation of a double pole in (2.3) below at $s = 1$ when γ equals a negative integer.

where $\omega(\sigma) = 0$ $(\sigma > 1)$, $\frac{1}{2} - \frac{1}{2}\sigma$ $(0 \le \sigma \le 1)$, $\frac{1}{2} - \sigma$ $(\sigma < 0)$ and $\alpha = 1$ $(0 \le \sigma \le 1)$, $\alpha = 0$ otherwise [3, p. 25], [11, p. 95]. The zeta function $\zeta(s)$ has a simple pole of unit residue at $s = 1$ and the evaluations for positive integer k

$$\zeta(0) = -\tfrac{1}{2}, \quad \zeta(-2k) = 0, \quad \zeta(1-2k) = -\frac{B_{2k}}{2k},$$
$$B_0 = 1,\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42}, \dots\ , \tag{1.5}$$

where B_k are the Bernoulli numbers. Finally, we have the well-known functional relation satisfied by $\zeta(s)$ given by

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s)\Gamma(1-s) \sin \tfrac{1}{2}\pi s. \tag{1.6}$$

2. Asymptotic expansion for $|a| \to \infty$ in $|\arg a| < \frac{1}{2}\pi$

We first consider the generalised Mathieu series $S_{\mu,\gamma}(a;b)$ defined in (1.3) by

$$S_{\mu,\gamma}(a;b) = a^{-\delta} e^{-b} \sum_{n=1}^{\infty} h(n/a), \qquad h(x) := \frac{x^\gamma}{(1+x^2)^\mu} \exp\left[\frac{bx^2}{1+x^2}\right]. \tag{2.1}$$

We shall employ a Mellin transform approach as discussed in [9, Section 4.1.1]. The Mellin transform of $h(x)$ is $H(s) = \int_0^\infty x^{s-1} h(x)\,dx$, where

$$\begin{aligned} H(s) &= \int_0^\infty \frac{x^{\gamma+s-1}}{(1+x^2)^\mu} \exp\left[\frac{bx^2}{1+x^2}\right] dx = \frac{1}{2} \sum_{r=0}^{\infty} \frac{b^r}{r!} \int_0^\infty \frac{\tau^{\frac{1}{2}\gamma + \frac{1}{2}s + r - 1}}{(1+\tau)^{\mu+r}} d\tau \\ &= \frac{1}{2} \sum_{r=0}^{\infty} \frac{b^r}{r!} \frac{\Gamma(\frac{1}{2}\gamma + \frac{1}{2}s + r)\Gamma(\mu - \frac{1}{2}\gamma - \frac{1}{2}s)}{\Gamma(\mu + r)} \\ &= \frac{\Gamma(\frac{1}{2}\gamma + \frac{1}{2}s)\Gamma(\mu - \frac{1}{2}\gamma - \frac{1}{2})}{2\Gamma(\mu)} F(s), \quad F(s) := {}_1F_1(\tfrac{1}{2}(\gamma+s); \mu; b) \end{aligned} \tag{2.2}$$

in the strip $-\gamma < \Re(s) < \delta$. The evaluation of the integral has been carried out by the beta function and ${}_1F_1$ denotes the confluent hypergeometric function. Using the Mellin inversion theorem (see [9, p. 118]) we find

$$\sum_{n=1}^{\infty} h(n/a) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{c-\infty i}^{c+\infty i} H(s)(n/a)^{-s} ds = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} H(s)\zeta(s) a^s ds,$$

where $1 < c < \delta$ and $\zeta(s)$ is the Riemann zeta function. The inversion of the order of summation and integration is justified by absolute convergence provided $1 < c < \delta$. Then, from (2.1),

$$S_{\mu,\gamma}(a;b) = \frac{a^{-\delta} e^{-b}}{2\Gamma(\mu)} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Gamma(\tfrac{1}{2}\gamma + \tfrac{1}{2}s)\Gamma(\mu - \tfrac{1}{2}\gamma - \tfrac{1}{2}s)\, F(s)\, \zeta(s) a^s ds, \tag{2.3}$$

where $1 < c < \delta$.

From [4, Theorem 2.3] we have for finite values of μ and $b \neq 0$

$$F(s) = \frac{\Gamma(\mu)}{\sqrt{\pi}} e^{b/2} (\kappa b)^{\frac{1}{4}-\frac{1}{2}\mu} \cos(2\sqrt{\kappa b} - \tfrac{1}{2}\pi b + \tfrac{1}{4}\pi)\{1 + O(|\kappa|^{-\frac{1}{2}})\}$$

as $|\kappa| \to \infty$ in $-\pi < \arg \kappa \leq \pi$, where $\kappa := \frac{1}{2}(\mu - \gamma - s)$. It then follows that with $s = \sigma + it$, where σ and t are real,

$$F(s) = O(|t|^{\frac{1}{4}-\frac{1}{2}\mu} \exp[\sqrt{|bt|}\,]), \qquad t \to \pm\infty. \tag{2.4}$$

From the estimates in (1.4), the integral in (2.3) then defines $S_{\mu,\gamma}(a;b)$ for complex a in the sector $|\arg a| < \frac{1}{2}\pi$.

Since ${}_1F_1(A;B;z)$ is entire in the parameter A [7, p. 322], the integrand in (2.3) has a simple pole at $s = 1$ and infinite strings of simple poles at $s = -\gamma - 2k$ and $s = 2\mu - \gamma + 2k$, $k = 0, 1, 2, \ldots$. Let N be a positive integer and let $c' = 2N + \gamma - 1$. We consider the integral in (2.3) taken round the rectangular contour with vertices at $c \pm iT$ and $-c' \pm iT$, where $T > 0$, so that the side in $\Re(s) < 0$ parallel to the imaginary s-axis passes midway between the poles at $s = -2(N-1) - \gamma$ and $s = -2N - \gamma$. The contribution from the upper and lower sides of the rectangle $s = \sigma \pm iT$, $-c' \leq \sigma \leq c$, vanishes as $T \to \infty$ provided $|\arg a| < \frac{1}{2}\pi$, since, from (1.4) and (2.4), the modulus of the integrand is controlled by $O(T^{\omega(\sigma)+\frac{1}{2}\mu-\frac{3}{4}} \log T \exp[-\Delta T + \sqrt{|b|T}])$, where $\Delta = \frac{1}{2}\pi - |\arg a|$. Evaluation of the residues[2] at $s = 1$ and $s = -\gamma - 2k$ $(0 \leq k \leq N-1)$ then yields

$$S_{\mu,\gamma}(a;b) = \frac{\Gamma(\frac{1}{2}\gamma + \frac{1}{2})\Gamma(\mu - \frac{1}{2}\gamma - \frac{1}{2})\, F(1)}{2e^b \Gamma(\mu) a^{2\mu-\gamma-1}} + \frac{e^{-b}}{\Gamma(\mu)} \sum_{k=0}^{N-1} \frac{(-1)^k \Gamma(\mu + k)}{k!\, a^{2(k+\mu)}}\, \zeta(-\gamma - 2k)\, C_k + R_N(a),$$

where the coefficients C_k are defined by

$$C_k = {}_1F_1(-k;\mu;b) \qquad k = 0, 1, 2, \ldots\ . \tag{2.5}$$

The C_k are consequently polynomials in b of degree k and

$$C_0 = 1, \quad C_1 = 1 - \frac{b}{\mu}, \quad C_2 = 1 - \frac{2b}{\mu} + \frac{b^2}{\mu(\mu+1)}, \ldots\ .$$

The remainder integral $R_N(a)$ is given by

$$\begin{aligned} R_N(a) &= \frac{a^{-\delta} e^{-b}}{2\Gamma(\mu)} \frac{1}{2\pi i} \int_{-c'-\infty i}^{-c'+\infty i} \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\gamma)\Gamma(\mu - \tfrac{1}{2}\gamma - \tfrac{1}{2}s)\, F(s)\, \zeta(s) a^s ds \\ &= \frac{a^{-\delta} e^{-b}}{4\Gamma(\mu)} \frac{1}{2\pi i} \int_{-c'-\infty i}^{-c'+\infty i} \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\gamma)\Gamma(\mu - \frac{1}{2}\gamma - \frac{1}{2}s)}{\Gamma(s)\, \cos\frac{1}{2}\pi s\ (2\pi a)^{-s}}\, F(s)\, \zeta(1-s) ds. \end{aligned}$$

[2]The residue of $\Gamma(s)$ at $s = -k$ is $(-1)^k/k!$.

Here we have used the functional relation in (1.6) and the reflection formula for the gamma function. If we set $s = -c' + it$ and use the fact that $|\zeta(\sigma + it)| \leq \zeta(\sigma)$ when $\sigma > 1$, we find

$$|R_N(a)| \leq \frac{(2\pi)^{-2N-\gamma}\zeta(2N+\gamma)}{4e^b\Gamma(\mu)|a|^{2N+2\mu-1}} \int_{-\infty}^{\infty} g(t)\,|F(-c'+it)|\,e^{\theta t - \frac{1}{2}\pi|t|}\,dt,$$

where

$$g(t) = \left|\frac{\Gamma(-N+\frac{1}{2}+\frac{1}{2}it)\Gamma(\mu+N-\frac{1}{2}-\frac{1}{2}it)e^{\frac{1}{2}\pi|t|}}{\Gamma(-c'+it)\cos\frac{1}{2}\pi(-c'+it)}\right|$$

and $\theta = \arg a$. Since $g(t) = O(|t|^{\mu+c'-\frac{1}{2}})$ as $t \to \pm\infty$, by the first estimate in (1.4), it follows from (2.4) that the above integral is independent of $|a|$ and converges when $|\arg a| < \frac{1}{2}\pi$. Hence $R_N(a) = O(|a|^{-2N-2\mu+1})$ and we consequently obtain the expansion:

Theorem 1. *With the conditions on the parameters μ, γ given in (1.4), we have the asymptotic expansion*

$$S_{\mu,\gamma}(a;b) \sim \frac{\Gamma(\frac{1}{2}\gamma+\frac{1}{2})\Gamma(\mu-\frac{1}{2}\gamma-\frac{1}{2})\,F(1)}{2e^b\Gamma(\mu)a^{2\mu-\gamma-1}} + \frac{e^{-b}}{\Gamma(\mu)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\mu+k)}{k!\,a^{2(k+\mu)}}\,\zeta(-\gamma-2k)\,C_k \qquad (2.6)$$

for $|a| \to \infty$ in the sector $|\arg a| < \frac{1}{2}\pi$. The quantity $F(1)$ is defined in (2.2) and the coefficients C_k in (2.5).

The expansion of the alternating series $\tilde{S}_{\mu,\gamma}(a;b)$ can be deduced from that for $S_{\mu,\gamma}(a;b)$ in (2.6) by observing that for $\delta > 1$

$$\tilde{S}_{\mu,\gamma}(a;b) = S_{\mu,\gamma}(a;b) - 2^{1-\delta}S_{\mu,\gamma}(\tfrac{1}{2}a;b).$$

Then, from (2.6), we obtain the following result.

Theorem 2. *With the conditions on the parameters μ, γ given in (1.4), we have the asymptotic expansion when $\delta > 1$*

$$\tilde{S}_{\mu,\gamma}(a;b) \sim \frac{e^{-b}}{\Gamma(\mu)}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(\mu+k)}{k!\,a^{2k+2\mu}}\,(2^{2k+\gamma+1}-1)\,\zeta(-\gamma-2k)\,C_k \qquad (2.7)$$

for $|a| \to \infty$ in the sector $|\arg a| < \frac{1}{2}\pi$, where the coefficients C_k are specified in (2.5).

In the case of the standard Mathieu series in (1.1) and (1.2), with $\mu = 2$, $\gamma = 1$, $b = 0$, we find from (2.6) and (2.7) that

$$S_2(a) \sim \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k B_{2k}}{a^{2k+2}}, \qquad \tilde{S}_2(a) \sim \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{a^{2k+2}}(2^{2k}-1)B_{2k}$$

as $|a| \to \infty$ in $|\arg a| < \frac{1}{2}\pi$, which agree with previous results[3] for the case $a \to +\infty$ [1, 10]. In these last expansions we have replaced $\zeta(-1-2k)$ by its representation in terms of the Bernoulli numbers by (1.5) and made the change of summation index $k \to k-1$.

The above procedure can be extended without difficulty to the more general functional series

$$S_{\mu,\gamma,\lambda}(a;b) = \sum_{n=1}^{\infty} \frac{n^{\gamma}}{(n^{\lambda}+a^{\lambda})^{\mu}} \exp\left[\frac{-a^{\lambda}b}{n^{\lambda}+a^{\lambda}}\right] \quad (\mu > 0,\ \lambda > 0,\ \gamma > -1,\ b \in \mathbf{R}) \tag{2.8}$$

and its alternating variant $\tilde{S}_{\mu,\gamma,\lambda}(a;b)$, where it is supposed that $\delta := \lambda\mu - \gamma > 1$ for absolute convergence of both series. For complex a it is sufficient to confine our attention to the sector $|\arg a| \leq \pi/\lambda$; if λ is non-integer then a branch-point structure is introduced. Then, for $1 < c < \delta$,

$$S_{\mu,\gamma,\lambda}(a;b) = \frac{a^{-\delta}e^{-b}}{\lambda\Gamma(\mu)} \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \Gamma(\frac{\gamma+s}{\lambda})\Gamma(\mu - \frac{\gamma+s}{\lambda})\, F_{\lambda}(s)\, \zeta(s)\, a^{s} ds,$$

$$F_{\lambda}(s) := {}_1F_1(\tfrac{\gamma+s}{\lambda}; \mu; b) \tag{2.9}$$

and, omitting the details, we find the following asymptotic expansions.

Theorem 3. *With the parameters defined in (2.8), we have the asymptotic expansions when* $\delta = \lambda\mu - \gamma > 1$

$$S_{\mu,\gamma,\lambda}(a;b) \sim \frac{\Gamma(\frac{\gamma+1}{\lambda})\Gamma(\mu - \frac{\gamma+1}{\lambda})\, F_{\lambda}(1)}{\lambda e^{b}\Gamma(\mu) a^{\lambda\mu-\gamma-1}} + \frac{e^{-b}}{\Gamma(\mu)} \sum_{k=0}^{\infty} \frac{(-1)^{k}\Gamma(\mu+k)}{k!\, a^{\lambda(k+\mu)}}\, \zeta(-\gamma-\lambda k)\, C_k \tag{2.10}$$

and

$$\tilde{S}_{\mu,\gamma,\lambda}(a;b) \sim \frac{e^{-b}}{\Gamma(\mu)} \sum_{k=0}^{\infty} \frac{(-1)^{k-1}\Gamma(\mu+k)}{k!\, a^{\lambda(k+\mu)}}\, (2^{\lambda k+\gamma+1}-1)\, \zeta(-\gamma-\lambda k)\, C_k \tag{2.11}$$

for $|a| \to \infty$ *in the sector* $|\arg a| < \pi/\lambda$. *The quantity* $F_{\lambda}(1)$ *and the coefficients* C_k *are defined in (2.9) and (2.5).*

In the case $b = 0$ the expansions in (2.10) and (2.11) agree with those derived in [12] for $a \to +\infty$. We note that when $\lambda = 2$, $\gamma = 1$ the expansions (2.10) and (2.11) reduce to (2.6) and (2.7), respectively. Also, in the case $\lambda = 1$, $\mu = 1$, $-1 < \gamma < 0$ and $b = 0$ the expansion of $S_{1,\gamma}(a;0)$ agrees with that given in [9, (4.2.3)] with $\alpha = \gamma + 1$, viz.

$$\sum_{n=1}^{\infty} \frac{n^{\alpha-1}}{n+a} \sim \frac{\pi a^{\alpha-1}}{\sin \pi\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^{k}}{a^{k+1}}\, \zeta(1-\alpha-k) \qquad (0 < \alpha < 1)$$

[3] There is a misprint in (3.6) of [10]; a factor $(-1)^{j}$ is missing from the sum on the left-hand side and the leading term should be $1/(4r^4)$.

as $|a| \to \infty$ in $|\arg a| < \pi$.

3. Numerical results

We present in Table 1 some numerical results to illustrate the accuracy of the expansions (2.6), (2.7), (2.10) and (2.11) for different parameters when $a = 10e^{i\theta}$ with $|\theta| < \pi/\lambda$. The values shown give the absolute relative error in the computation of the different generalised Mathieu series compared with high-precision evaluation of the infinite series.

$b = 0$ $\mu = 1.5,\ \lambda = 2,\ \gamma = 1$		$b = -0.5$ $\mu = 3,\ \lambda = 1.5,\ \gamma = 1$		$b = 0.5$ $\mu = 2,\ \lambda = 1,\ \gamma = -0.5$	
θ	Error	θ	Error	θ	Error
0	3.713×10^{-11}	0	9.105×10^{-11}	0	4.980×10^{-6}
0.25π	3.703×10^{-11}	0.30π	8.925×10^{-11}	0.50π	4.473×10^{-6}
0.40π	4.613×10^{-6}	0.50π	5.662×10^{-9}	0.75π	4.019×10^{-6}

$b = 0$ $\mu = 1.5,\ \lambda = 2,\ \gamma = 1$		$b = -0.5$ $\mu = 3,\ \lambda = 1.5,\ \gamma = 1$		$b = 0.5$ $\mu = 2,\ \lambda = 1,\ \gamma = -0.5$	
θ	Error	θ	Error	θ	Error
0	1.924×10^{-9}	0	1.030×10^{-7}	0	1.551×10^{-9}
0.25π	1.337×10^{-5}	0.30π	9.956×10^{-7}	0.50π	1.904×10^{-9}
0.30π	5.689×10^{-4}	0.40π	1.635×10^{-4}	0.75π	2.572×10^{-7}

Table 1: The absolute relative error in the computation of $S_{\mu,\gamma,\lambda}(a;b)$ (upper table) and $\tilde{S}_{\mu,\gamma,\lambda}(a;b)$ (lower table) using the asymptotic expansions (2.10) and (2.11) for different parameters when $a = 10\exp(i\theta)$. The truncation index in each case is $k = 10$.

The value of $\zeta(-\gamma - \lambda k)$ appearing in these expansions can be evaluated by means of (1.6) and, in the case of integer values of γ and λ by means of (1.5). The optimal truncation index of a (divergent) asymptotic expansion corresponds to truncation at, or near, the least term in magnitude. From (1.6) and the well-known fact that $\Gamma(k + a)/\Gamma(k + b) \sim k^{a-b}$ for $k \to +\infty$, the optimal truncation index k_0 for the asymptotic series in (2.10) is found to be

$$k_0 \simeq \frac{2\pi|a|}{\lambda},$$

with that for the series (2.11) being reduced by a factor of 2. It is then seen that with $|a| = 10$, the truncation index $k = 10$ employed in Table 1 is sub-optimal in each case.

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