

ON THE ASYMPTOTIC EXPANSION OF A BINOMIAL SUM INVOLVING POWERS OF THE SUMMATION INDEX

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Abstract. Work elsewhere [1, 3] has revealed the leading asymptotic behaviour of the binomial sum $S_p(n)$ defined by

$$S_p(n) = \sum_{j=1}^n j^p \binom{n+j}{j}$$

in the limit $n \rightarrow \infty$ in the case of positive integer p . In this paper, we establish the asymptotic expansion of $S_p(n)$ first for positive integer p and secondly, by means of an integral representation for the sum, for arbitrary values of the index p .

1. Introduction

Consideration of the binomial sum

$$S_p(n) = \sum_{j=1}^n j^p \binom{n+j}{j} \tag{1.1}$$

has been motivated by the recent study of a multi-link inverted pendulum enumeration problem [2]. The main properties of $S_p(n)$ are examined for positive integer p in [3], where its explicit evaluation for $1 \leq p \leq 5$ is given. It is shown among other things that the large- n behaviour is described by

$$S_p(n) \sim 2n^p \binom{2n}{n} \quad (n \rightarrow \infty). \tag{1.2}$$

Two alternative proofs of this result are to be given in a further paper [1]: the first uses an elaborate and lengthy application of the Euler-Maclaurin summation formula, and the second uses a straightforward decomposition of the sum in terms of Stirling numbers of the second kind. These proofs each differ significantly from that in [3].

In this paper we offer two derivations of the asymptotic expansion of $S_p(n)$ as $n \rightarrow \infty$. The first approach is valid for positive integer values of p and follows from the above-mentioned decomposition of $S_p(n)$ in terms of the Stirling numbers. The second approach is valid for arbitrary, finite values of p and uses an integral representation for $S_p(n)$ combined with the method of steepest descents.

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2. The expansion for positive integer p

Let $s(p, j)$ be the Stirling number of the second kind and $[x]_j$ denote (with $[x]_0 = 1$) the usual falling factorial function $[x]_j = x(x-1) \cdots (x-j+1)$. In the second proof in [1] it is established that

$$S_p(n) = \binom{2n}{n} \{2n^p - F(n, p)\}, \tag{2.1}$$

where, with $\alpha_j(p) = (2j+1)s(p, j)$ and $g(n; j) = [n]_j / (n+j+1)$,

$$F(n, p) = \sum_{j=0}^p \alpha_j(p)g(n; j). \tag{2.2}$$

Note that for large n we have $g(n; j) = O(n^{j-1})$.

In order to generate a series expansion for $S_p(n)$ it is necessary to extract terms within $F(n, p)$ of $O(n^{p-r})$ for $r = 1, 2, 3, \dots$. Before doing this, we note the following values of the Stirling number

$$\begin{aligned} s(p, p) &= 1, \\ s(p, p-1) &= p(p-1)/2, \\ s(p, p-2) &= p(p-1)(p-2)(3p-5)/24 \end{aligned}$$

and that

$$\begin{aligned} g(n; p) &= (n-1)(n-2) \cdots (n-p+1) \left\{ 1 - \frac{(p+1)}{n} + \frac{(p+1)^2}{n^2} - \frac{(p+1)^3}{n^3} + \dots \right\} \\ &= A(n, p)B(n, p), \end{aligned} \tag{2.3}$$

say, where $A(n, p) = [n-1]_{p-1}$ is a polynomial of degree $p-1$ and

$$B(n, p) = \sum_{r=0}^{\infty} (-)^r (p+1)^r n^{-r}$$

is a power series (each in n).

It is immediate¹ that $[n^{p-1}]\{g(n; p)\} = 1$, and so

$$[n^{p-1}]\{F(n, p)\} = \alpha_p(p)[n^{p-1}]\{g(n; p)\} = \alpha_p(p) = 2p+1. \tag{2.4}$$

To next order we have $[n^{p-2}]\{g(n; p-1)\} = 1$ trivially, and construct $[n^{p-2}]\{g(n; p)\}$ as

$$\begin{aligned} [n^{p-2}]\{g(n; p)\} &= [n^{p-1}]\{A(n, p)\} \cdot [n^{-1}]\{B(n, p)\} + [n^{p-2}]\{A(n, p)\} \cdot [n^0]\{B(n, p)\} \\ &= -(p+1) - p(p-1)/2 \\ &= -(p^2 + p + 2)/2, \end{aligned} \tag{2.5}$$

¹We employ the standard notation $[x^r]\{f(x)\}$ to denote the coefficient of x^r in the expansion of $f(x)$.

whence, with $\alpha_{p-1}(p) = (2p-1)s(p, p-1)$,

$$\begin{aligned} [n^{p-2}]\{F(n, p)\} &= \alpha_{p-1}(p)[n^{p-2}]\{g(n; p-1)\} + \alpha_p(p)[n^{p-2}]\{g(n; p)\} \\ &= p(p-1)(2p-1)/2 - (2p+1)(p^2+p+2)/2 \\ &= -(3p^2+2p+1). \end{aligned} \quad (2.6)$$

To obtain the $O(n^{p-3})$ term, we first observe that $[n^{p-3}]\{g(n; p-2)\} = 1$ (again trivially). Then, directly from (2.5),

$$[n^{p-3}]\{g(n; p-1)\} = -[(p-1)^2 + (p-1) + 2]/2 = -(p^2 - p + 2)/2, \quad (2.7)$$

and this in turn gives

$$\begin{aligned} [n^{p-3}]\{g(n; p)\} &= [n^{p-1}]\{A(n, p)\} \cdot [n^{-2}]\{B(n, p)\} + [n^{p-2}]\{A(n, p)\} \cdot [n^{-1}]\{B(n, p)\} \\ &\quad + [n^{p-3}]\{A(n, p)\} \cdot [n^0]\{B(n, p)\} \\ &= (p+1)^2 + p(p-1)(p+1)/2 + p(p-1)(p-2)(3p-1)/24 \\ &= (3p^4 + 2p^3 + 33p^2 + 34p + 24)/24. \end{aligned} \quad (2.8)$$

With $\alpha_{p-2}(p) = (2p-3)s(p, p-2)$, it then follows that

$$\begin{aligned} [n^{p-3}]\{F(n, p)\} &= \alpha_{p-2}(p)[n^{p-3}]\{g(n; p-2)\} + \alpha_{p-1}(p)[n^{p-3}]\{g(n; p-1)\} \\ &\quad + \alpha_p(p)[n^{p-3}]\{g(n; p)\} \\ &= p(p-1)(p-2)(2p-3)(3p-5)/24 \\ &\quad - p(p-1)(2p-1)(p^2-p+2)/4 \\ &\quad + (2p+1)(3p^4+2p^3+33p^2+34p+24)/24 \\ &= (26p^3+15p^2+25p+6)/6. \end{aligned} \quad (2.9)$$

Additional terms can be obtained by continuation of this procedure but this becomes increasingly laborious. From (2.4), (2.6) and (2.9) we write

$$\begin{aligned} F(n, p) &= (2p+1)n^{p-1} - (3p^2+2p+1)n^{p-2} \\ &\quad + (26p^3+15p^2+25p+6)n^{p-3}/6 + O(n^{p-4}) + \dots + O(1) + O(n^{-1}), \end{aligned}$$

so that (2.1) reads

$$\begin{aligned} S_p(n) &= 2n^p \binom{2n}{n} \left\{ 1 - \frac{(2p+1)}{2n} + \frac{(3p^2+2p+1)}{2n^2} \right. \\ &\quad \left. - \frac{(26p^3+15p^2+25p+6)}{12n^3} + O(n^{-4}) + \dots + O(n^{-p-1}) \right\}. \end{aligned} \quad (2.10)$$

Finally, we substitute into (2.10) the known result

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \dots \right) \quad (n \rightarrow \infty)$$

which, after a little algebra, then yields our desired expansion in the form

$$S_p(n) \sim \frac{2^{2n+1} n^p}{\sqrt{\pi n}} \left\{ 1 - \frac{(8p+5)}{8n} + \frac{(192p^2+144p+73)}{128n^2} - \frac{(6656p^3+4416p^2+6808p+1725)}{3072n^3} + \dots \right\} \quad (2.11)$$

as $n \rightarrow \infty$.

We remark that this formulation has the decomposition (2.1) as its basis and so is necessarily restricted to positive integer values of the index p . The method has relied on the extraction of individual terms within the polynomial $F(n, p)$ and combination with the asymptotic expansion for $\binom{2n}{n}$. Although some aspects of the procedure readily lend themselves to automation by computer algebra, it is required – in accordance with the number of terms in the expansion of $S_p(n)$ sought – to have ever more complicated closed-form expressions for the Stirling numbers $s(p, p-r)$ with $r \geq 3$.

3. The expansion for general values of p

The aim in this section is to derive the asymptotic expansion of $S_p(n)$ for arbitrary (finite) index p . The analysis is presented for real p (positive or negative), but is easily extended to complex values of p . We employ the integral representation

$$\frac{\Gamma(n+j+1)}{\Gamma(j+1)n!} = \frac{1}{2\pi i} \int_0^{(1+)} \frac{t^{n+j}}{(t-1)^{n+1}} dt,$$

where the integration path is a loop in the positive sense surrounding the point $t = 1$. This result is easily established by consideration of the residue of the integrand at the pole $t = 1$. Then, from (1.1), we find

$$\begin{aligned} S_p(n) &= \frac{1}{2\pi i} \int_0^{(1+)} \frac{t^n}{(t-1)^{n+1}} \sum_{j=1}^n j^p t^j dt \\ &= \frac{n^p}{2\pi i} \int_0^{(1+)} \frac{t^{2n}}{(t-1)^{n+1}} \sum_{k=0}^{n-1} t^{-k} \left(1 - \frac{k}{n}\right)^p dt. \end{aligned} \quad (3.1)$$

If we introduce the phase function $\psi(t)$ and the amplitude function $F(t)$ by

$$\psi(t) = 2 \log t - \log(t-1), \quad F(t) = \sum_{k=0}^{n-1} t^{-k} \left(1 - \frac{k}{n}\right)^p, \quad (3.2)$$

then we can write

$$S_p(n) = \frac{n^p}{2\pi i} \int_0^{(1+)} \frac{e^{n\psi(t)}}{t-1} F(t) dt. \quad (3.3)$$

The exponential factor in the integrand in (3.3) has a simple saddle point at $t = 2$ (where $\psi'(t) = 0$). The steepest descent path through the saddle point is given by $\text{Im } \psi(t) = 0$, or (with $t = x + iy$)

$$2 \arctan \left(\frac{y}{x} \right) = \arctan \left(\frac{y}{x-1} \right).$$

This is readily shown to be the path $(x - 1)^2 + y^2 = 1$ and so is a circle of unit radius centred at $t = 1$. Let the integration path be as shown in Fig. 1. This path consists of part of the imaginary axis AA' between $\pm i$, the horizontal segments AB and $A'B'$ connecting $\pm i$ to $1 \pm i$, respectively, together with the semi-circle C passing through the saddle and the points $1 \pm i$ (part of the steepest descent path).

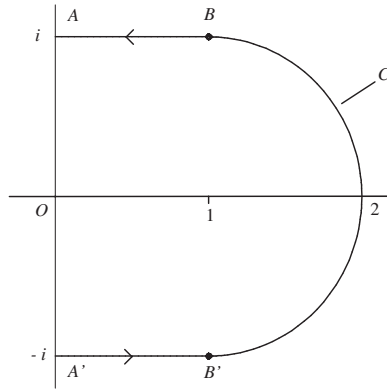


Figure 1: The integration path in the t -plane. The saddle point is at $t = 2$.

Let us denote the contributions to the first integral in (3.1) from the different portions of the path by $I_{AA'}$, I_{AB} , $I_{A'B'}$ and I_C . On the path AA' , we put $t = iu$, $-1 \leq u \leq 1$ to find

$$\begin{aligned} |I_{AA'}| &= \left| \int_{-i}^i \left(\frac{t}{t-1} \right)^n \sum_{j=1}^n j^p t^j \frac{dt}{t-1} \right| \leq 2 \int_0^1 \left(\frac{u}{\sqrt{1+u^2}} \right)^n \sum_{j=1}^n j^p u^j \frac{du}{\sqrt{1+u^2}} \\ &\leq 2^{1-\frac{1}{2}n} \sum_{j=1}^n j^p \int_0^1 u^j du \leq 2^{1-\frac{1}{2}n} \sum_{j=1}^n j^{p-1} \leq 2^{1-\frac{1}{2}n} n^\delta, \end{aligned}$$

where $\delta = p$ ($p > 1$), $\delta = 1$ ($p \leq 1$). Similarly, on the path AB we put $t = i + u$, $0 \leq u \leq 1$ to find

$$|I_{AB}| = \left| \int_i^{1+i} \left(\frac{t}{t-1} \right)^n \sum_{j=1}^n j^p t^j \frac{dt}{t-1} \right| \leq \int_0^1 \left(\frac{\sqrt{1+u^2}}{\sqrt{1+(1-u)^2}} \right)^n \sum_{j=1}^n \frac{j^p (1+u^2)^{j/2}}{\sqrt{1+(1-u)^2}} du$$

$$\begin{aligned} &\leq 2^{\frac{1}{2}n} \sum_{j=1}^n j^p \int_0^1 (1+u^2)^{j/2} du \leq 2^{\frac{1}{2}n} \sum_{j=1}^n j^p \int_0^1 (1+u)^{j/2} du \\ &\leq 2^{\frac{1}{2}n} \sum_{j=1}^n j^{p-1} 2^{\frac{1}{2}j+2} \leq 2^{n+2} \sum_{j=1}^n j^{p-1} \leq 2^{n+2} n^\delta. \end{aligned}$$

A similar estimate applies for the path $A'B'$. As the saddle point contribution is $O(e^{n\psi(2)}) = O(2^{2n})$ as $n \rightarrow \infty$, it is seen that the contributions from the rectilinear parts of the integration path are subdominant in this limit, and hence that the dominant contribution to $S_p(n)$ arises from the path C through the saddle point.

We now consider the contribution

$$I_C = \frac{n^p}{2\pi i} \int_C \frac{e^{n\psi(t)}}{t-1} F(t) dt, \tag{3.4}$$

where on the path C we have uniformly $|t| > 1$. In order to deal with $F(t)$ we shall require the following lemma.

LEMMA 1. Let $\Theta \equiv td/dt$ and $\sigma = t/(t-1)$. Then, for $|t| > 1$ and non-negative integer r , we have

$$\sum_{k=0}^{n-1} k^r t^{-k} = (-)^r \Theta^r \sigma + O(n^r t^{-n}) \tag{3.5}$$

as $n \rightarrow \infty$.

Proof. We first observe, by differentiation of the geometric series, that $\sum_{k=0}^\infty k^r t^{-k} = (-)^r \Theta^r \sigma$ when $|t| > 1$. Then, since $|t| > 1$,

$$\sum_{k=0}^{n-1} k^r t^{-k} = \left(\sum_{k=0}^\infty - \sum_{k=n}^\infty \right) k^r t^{-k} = (-)^r \Theta^r \sigma - t^{-n} \sum_{j=0}^\infty (n+j)^r t^{-j},$$

where

$$\begin{aligned} t^{-n} \sum_{j=0}^\infty (n+j)^r t^{-j} &= t^{-n} \sum_{k=0}^r \binom{r}{k} n^{r-k} \sum_{j=0}^\infty j^k t^{-j} = t^{-n} \sum_{k=0}^r (-)^k \binom{r}{k} n^{r-k} \Theta^k \sigma \\ &= t^{-n} (n - \Theta)^r \sigma. \end{aligned}$$

It then follows that

$$\sum_{k=0}^{n-1} k^r t^{-k} = (-)^r \Theta^r \sigma - t^{-n} (n - \Theta)^r \sigma = (-)^r \Theta^r \sigma + O(n^r t^{-n})$$

as $n \rightarrow \infty$, thereby establishing the lemma. \square

Let N be a positive integer. Application of the binomial theorem to write $F(t)$ in (3.2) in the form

$$F(t) = \sum_{j=0}^\infty \frac{(-)^j}{n^j} \binom{p}{j} \sum_{k=0}^{n-1} k^j t^{-k},$$

shows that, provided $|t| > 1$,

$$F(t) = \sum_{j=0}^{N-1} \frac{(-)^j}{n^j} \binom{p}{j} \sum_{k=0}^{n-1} k^j t^{-k} + O(n^{-N}) \Theta^N \sigma = \sum_{j=0}^{N-1} \binom{p}{j} \frac{\Theta^j \sigma}{n^j} + O(t^{-n}) + O(n^{-N}) \Theta^N \sigma$$

by (3.5). Insertion of the above representation into (3.4) then yields

$$I_C = n^p \left\{ \sum_{r=0}^{N-1} \binom{p}{r} \frac{J_r}{n^r} + R_N \right\},$$

where

$$J_r = \frac{1}{2\pi i} \int_C \frac{e^{n\psi(t)}}{t-1} \Theta^r \sigma dt$$

and

$$R_N = \frac{1}{2\pi i} \int_C \frac{e^{n\psi(t)}}{t-1} \{O(t^{-n}) + \Theta^N \sigma O(n^{-N})\} dt = J_N O(n^{-N})$$

upon absorbing the exponentially subdominant contribution resulting from $O(t^{-n})$ into the $O(n^{-N})$ term.

The asymptotic expansion of the integrals J_r is discussed in the Appendix. From (A.5), we have for positive integers M_r

$$J_r = \frac{2^{2n+1} (-)^r}{\sqrt{\pi n}} \left\{ \sum_{j=0}^{M_r-1} \frac{(-)^j A_j^{(r)}}{n^j} + O(n^{-M_r}) \right\}$$

for large n , where the coefficients $A_j^{(r)}$ for $r+j \leq 5$ are given in Table 2. If we set the index $M_r = N-r$ ($0 \leq r \leq N-1$), we obtain

$$\begin{aligned} I_C &= \frac{2^{2n+1} n^p}{\sqrt{\pi n}} \left\{ \sum_{r=0}^{N-1} \frac{(-)^r}{n^r} \binom{p}{r} \left\{ \sum_{j=0}^{N-r} \frac{(-)^j A_j^{(r)}}{n^j} + O(n^{-N+r}) \right\} + O(n^{-N}) \right\} \\ &= \frac{2^{2n+1} n^p}{\sqrt{\pi n}} \left\{ \sum_{k=0}^{N-1} \frac{(-)^k}{n^k} \sum_{r=0}^k \binom{p}{r} A_{k-r}^{(r)} + O(n^{-N}) \right\} \end{aligned} \tag{3.6}$$

upon setting $k = r + j$ and summing the double sum diagonally.

Then, from (3.4) and (3.6), the dominant contribution to $S_p(n)$ takes the form

$$S_p(n) \sim \frac{2^{2n+1} n^p}{\sqrt{\pi n}} \sum_{k=0}^{\infty} \frac{(-)^k c_k}{n^k} \quad (n \rightarrow \infty) \tag{3.7}$$

valid for arbitrary, finite values of p , where the coefficients c_k are defined by

$$c_k = \sum_{r=0}^k \binom{p}{r} A_{k-r}^{(r)}.$$

Use of the values of the coefficients $A_j^{(r)}$ in Table 2 shows that the c_k for $k \leq 5$ have the values

$$\begin{aligned}
 c_0 &= 1, & c_1 &= \frac{1}{8}(8p + 5), & c_2 &= \frac{1}{128}(192p^2 + 144p + 73), \\
 c_3 &= \frac{1}{3072}(6656p^3 + 4416p^2 + 6808p + 1725), \\
 c_4 &= \frac{1}{32768}(102400p^4 + 30720p^3 + 214400p^2 + 52320p + 18459), \\
 c_5 &= \frac{1}{3932160}(17727488p^5 - 6082560p^4 + 70517760p^3 - 5950080p^2 \\
 &\qquad\qquad\qquad + 21964072p + 2222325).
 \end{aligned}$$

It is seen that the coefficients up to and including c_3 agree with those obtained in Section 2 valid for positive integer p . We note the presence of negative terms in the coefficient c_5 .

We present the results of numerical calculations. In Table 1 we show the absolute value of the relative error in the computation of the sum $S_p(n)$ by means of the asymptotic expansion (3.7) for different values of n , p and truncation index j . In each case the value of n has been chosen so that the optimal truncation point of the asymptotic series in (3.7) corresponds to $j > 5$. Consequently, the relative error progressively decreases with each increment in the truncation index, thereby confirming the validity of the asymptotic series (3.7).

| j | $n = 50$ | | $n = 100$ | |
|-----|------------------------|------------------------|-------------------------|-------------------------|
| | $p = -\frac{1}{2}$ | $p = \frac{3}{2}$ | $p = \frac{1}{2}$ | $p = \frac{3}{4}$ |
| 0 | 2.348×10^{-3} | 4.205×10^{-2} | 1.123×10^{-2} | 1.371×10^{-2} |
| 1 | 1.574×10^{-4} | 2.233×10^{-3} | 1.502×10^{-4} | 2.249×10^{-4} |
| 2 | 3.928×10^{-6} | 1.145×10^{-4} | 2.293×10^{-6} | 3.932×10^{-6} |
| 3 | 2.550×10^{-7} | 5.819×10^{-6} | 3.294×10^{-8} | 6.799×10^{-8} |
| 4 | 1.786×10^{-8} | 2.936×10^{-7} | 5.256×10^{-10} | 1.201×10^{-9} |
| 5 | 1.544×10^{-9} | 1.477×10^{-8} | 6.841×10^{-12} | 2.061×10^{-11} |

Table 1: The relative error in the computation of $S_p(n)$ by (3.7) for different truncation index j .

4. Concluding remarks

We have presented two methods to generate an asymptotic expansion for the sum $S_p(n)$ in (1.1) for large n , completing an examination of properties of the sum discussed elsewhere. The first method is valid for positive integer p and relies on a decomposition of $S_p(n)$ in terms of the Stirling numbers of the second kind. The second method is valid for arbitrary p and is based on an integral representation combined with an application of the method of steepest descents.

When $p = 0$, we have the well-known result [4, p. 619]

$$S_0(n) + 1 = \sum_{j=0}^n \binom{n+j}{j} = \binom{2n+1}{n} = \frac{2^{2n+1}}{\sqrt{\pi}} \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n+2)}.$$

By means of the expansion of the ratio of two gamma functions [5, p. 119], [6, p. 50] this produces the large- n expansion

$$S_0(n) \sim \frac{2^{2n+1}}{\sqrt{\pi n}} \left\{ 1 - \frac{5}{8n} + \frac{73}{128n^2} - \frac{575}{1024n^3} + \frac{18459}{32768n^4} - \frac{148155}{262144n^5} + \dots \right\}.$$

This is seen to be in agreement with (3.7) when the coefficients c_k are evaluated at $p = 0$.

Appendix A: The expansion of the integrals J_r

We consider the asymptotic expansion of the integrals

$$J_r = \frac{1}{2\pi i} \int_C \frac{e^{n\psi(t)}}{t-1} \Theta^r \sigma dt \quad (r = 0, 1, 2, \dots) \tag{A.1}$$

for $n \rightarrow \infty$, where $\psi(t)$ is defined in (3.2), C is the semi-circular path described in Section 3 that passes through the saddle point $t = 2$, $\Theta \equiv td/dt$ and $\sigma = t/(t-1)$. Routine calculations show that

$$\Theta^r \sigma = \frac{(-)^r t q_r(t)}{(t-1)^{r+1}},$$

where

$$q_0(t) = q_1(t) = 1, \quad q_2(t) = t + 1, \quad q_3(t) = t^2 + 4t + 1, \\ q_4(t) = t^3 + 11t^2 + 11t + 1, \quad q_5(t) = t^4 + 26t^3 + 66t^2 + 26t + 1, \dots$$

We now make the change of variable $t \mapsto \tau$ in (A.1) given by

$$-\tau^2 = \psi(t) - 2 \log 2 = \frac{(t-2)^2}{4} - \frac{(t-2)^3}{4} + \frac{7(t-2)^4}{32} + \dots,$$

so that the point $t = 2$ corresponds to $\tau = 0$. Straightforward differentiation gives

$$\frac{dt}{d\tau} = -\frac{2\tau t(t-1)}{t-2}$$

and inversion of the above series yields

$$t - 2 = 2i\tau - 2\tau^2 - \frac{3}{2}i\tau^3 + \tau^4 + \frac{29i\tau^5}{48} - \frac{\tau^6}{3} - \frac{11i\tau^7}{64} + \dots \tag{A.2}$$

Then, since the endpoints of C at $t = 1 \pm i$ correspond to $\tau = \tau_0 = \pm(\log 2)^{1/2}$, we find

$$J_r = \frac{2^{2n}}{2\pi i} \int_{-\tau_0}^{\tau_0} \frac{e^{-n\tau^2}}{t-1} \Theta^r \sigma \frac{dt}{d\tau} d\tau = \frac{2^{2n+1}}{\pi} \int_{-\tau_0}^{\tau_0} e^{-n\tau^2} Q_r(\tau) d\tau, \tag{A.3}$$

where

$$Q_r(\tau) = \frac{(-)^r i \tau t^2 q_r(t)}{2(t-2)(t-1)^{r+1}}. \tag{A.4}$$

We present only the calculation of the expansion of J_0 , the details for J_r with $r = 1, 2, \dots$ being similar. With the help of *Mathematica* and (A.2), we obtain the expansion of $Q_0(\tau)$ in powers of τ given by²

$$Q_0(\tau) = \frac{i \tau t^2}{2(t-1)(t-2)} \stackrel{\text{e}}{=} \sum_{k=0}^{\infty} (-)^k a_k \tau^{2k},$$

where

$$a_0 = 1, \quad a_1 = \frac{5}{4}, \quad a_2 = \frac{73}{96}, \quad a_3 = \frac{115}{384}, \quad a_4 = \frac{879}{10240}, \quad a_5 = \frac{1411}{73728}, \dots$$

The limits of integration in (A.3) may be extended to $\pm\infty$ (thereby describing the full steepest descent path in the t -plane) with the introduction of an exponentially small error of $O(2^{-n})$. Neglecting exponentially small terms, we then obtain

$$J_0 \simeq \frac{2^{2n+1}}{\pi} \int_{-\infty}^{\infty} e^{-n\tau^2} Q_0(\tau) d\tau \sim \frac{2^{2n+1}}{\sqrt{\pi n}} \sum_{k=0}^{\infty} \frac{(-)^k a_k \Gamma(k + \frac{1}{2})}{n^k \Gamma(\frac{1}{2})} \quad (n \rightarrow \infty)$$

upon straightforward evaluation of the integrals in terms of the gamma function.

By means of similar calculations the expansions of $Q_r(\tau)$ for $1 \leq r \leq 5$ are found to be

$$\begin{aligned} Q_1(\tau) &\stackrel{\text{e}}{=} 1 - \frac{21}{4} \tau^2 + \frac{745}{96} \tau^4 - \frac{833}{128} \tau^6 + \frac{38959}{10240} \tau^8 - \dots, \\ Q_2(\tau) &\stackrel{\text{e}}{=} 3 - \frac{127}{4} \tau^2 + \frac{2665}{32} \tau^4 - \frac{44681}{384} \tau^6 + \dots, \\ Q_3(\tau) &\stackrel{\text{e}}{=} 13 - \frac{945}{4} \tau^2 + \frac{93205}{96} \tau^4 - \dots, \\ Q_4(\tau) &\stackrel{\text{e}}{=} 75 - \frac{8359}{4} \tau^2 + \dots, \\ Q_5(\tau) &\stackrel{\text{e}}{=} 541 + O(\tau^2). \end{aligned}$$

An analogous procedure then yields the desired expansion of J_r in the form

$$J_r \sim \frac{2^{2n+1} (-)^r}{\sqrt{\pi n}} \sum_{j=0}^{\infty} (-)^j A_j^{(r)} n^{-j} \tag{A.5}$$

as $n \rightarrow \infty$, where the coefficients $A_j^{(r)}$ for $r + j \leq 5$ are listed in Table 2. From (A.2) and (A.4), it is seen that $A_0^{(r)} = Q_r(0) = q_r(2)$.

²We use the symbol $\stackrel{\text{e}}{=}$ to indicate that only *even* powers of τ are included. The terms involving odd powers of τ do not enter into our calculations and so are not shown.

| $r \setminus j$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----------------|-----|------------------|---------------------|-----------------------|------------------------|-------------------------|
| 0 | 1 | $\frac{5}{8}$ | $\frac{73}{128}$ | $\frac{575}{1024}$ | $\frac{18459}{32768}$ | $\frac{148155}{262144}$ |
| 1 | 1 | $\frac{21}{8}$ | $\frac{745}{128}$ | $\frac{12495}{1024}$ | $\frac{818139}{32768}$ | |
| 2 | 3 | $\frac{127}{8}$ | $\frac{7995}{128}$ | $\frac{223405}{1024}$ | | |
| 3 | 13 | $\frac{945}{8}$ | $\frac{93205}{128}$ | | | |
| 4 | 75 | $\frac{8359}{8}$ | | | | |
| 5 | 541 | | | | | |

Table 2: The coefficients $A_j^{(r)}$ for $r+j \leq 5$.

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