

An extension of Saalschütz's summation theorem for the series ${}_{r+3}F_{r+2}$

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Abstract

The aim in this research note is to provide an extension of Saalschütz's summation theorem for the series ${}_{r+3}F_{r+2}(1)$ when r pairs of numeratorial and denominatorial parameters differ by positive integers. The result is obtained by exploiting a generalization of an Euler-type transformation recently derived by Miller and Paris [9].

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1. Introduction

The generalized hypergeometric function ${}_pF_q(x)$ may be defined for complex parameters and argument by the series

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{x^k}{k!}. \quad (1.1)$$

When $q = p$ this series converges for $|x| < \infty$, but when $q = p - 1$ convergence occurs when $|x| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. In (1.1) the Pochhammer symbol or ascending factorial $(a)_n$ is given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0) \\ a(a+1)\dots(a+n-1) & (n \in \mathbf{N}), \end{cases}$$

where Γ is the gamma function. The parametric excess s of the above series is defined by

$$s = \sum_{r=1}^q b_r - \sum_{r=1}^p a_r.$$

We shall adopt the convention of writing the finite sequence of parameters (a_1, \dots, a_p) simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k \equiv (a_1)_k \dots (a_p)_k,$$

where an empty product $p = 0$ reduces to unity. It is evident that whenever generalized hypergeometric functions of special argument reduce to Gamma functions the results are of considerable importance in applications. Until 1990, only a few classical summation theorems for ${}_2F_1$, ${}_3F_2$ and for higher order series were known. Subsequently, some progress has been made in generalizing these classical summation theorems; see [1, 3, 4, 5, 7, 10, 13, 15].

In our present investigation we shall be concerned with the following summation theorem due to Saalschütz [14, p. 49]

$${}_3F_2 \left(\begin{matrix} -n, a, b \\ c, 1+a+b-c-n \end{matrix} \middle| 1 \right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \quad (1.2)$$

where n is a nonnegative integer. Such a series is said to be Saalschützian since the parametric excess $s = 1$. As shown in [14, p. 49], this follows from taking the well-known Euler transformation

$$(1-x)^{a+b-c} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = {}_2F_1 \left(\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| x \right) \quad (1.3)$$

and equating coefficients of x^n on both sides of the equation. An extension of Saalschütz's theorem has been considered recently by Rakha and Rathie in [13] who showed that

$${}_4F_3 \left(\begin{matrix} -n, a, b, f+1 \\ c, 2+a+b-c-n, f \end{matrix} \middle| 1 \right) = \frac{(c-a-1)_n (c-b-1)_n (\eta+1)_n}{(c)_n (c-a-b-1)_n (\eta)_n},$$

where

$$\eta = \frac{(c-a-1)(c-b-1)f}{ab + (c-a-b-1)f}.$$

As an application of this result these authors also established the quadratic transformation

$$(1+x)^{-2a} {}_3F_2 \left(\begin{matrix} a, b, f+1 \\ a+b+\frac{3}{2}, f \end{matrix} \middle| \frac{4x}{(1+x)^2} \right) = {}_4F_3 \left(\begin{matrix} 2a, a-b-\frac{1}{2}, \omega_1+1, \omega_2+1 \\ a+b+\frac{3}{2}, \omega_1, \omega_2 \end{matrix} \middle| -x \right), \quad (1.4)$$

where

$$\omega_{1,2} = a \pm [a^2 + (a + \frac{1}{2})(a+b - \frac{1}{2})f / (b-f)]^{1/2}.$$

In [9], a generalization of the Euler transformation (1.3), when r pairs of numeratorial and denominatorial parameters differ by positive integers (m_r) , was obtained in the form

$$\begin{aligned} & {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| x \right) \\ &= (1-x)^{c-a-b-m} {}_{m+2}F_{m+1} \left(\begin{matrix} c-a-m, c-b-m, (\eta_m+1) \\ c, (\eta_m) \end{matrix} \middle| x \right) \end{aligned} \quad (1.5)$$

when $|x| < 1$. Here (η_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m \equiv m_1 + \dots + m_r$ given by

$$Q_m(t) = \sum_{k=0}^m B_k(a)_k (b)_k (t)_k G_{m,k}(t) \quad (1.6)$$

with

$$B_k \equiv (-1)^k A_k (c - a - m + k)_{m-k} (c - b - m + k)_{m-k}$$

and

$$G_{m,k}(t) \equiv {}_3F_2 \left(\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix} \middle| 1 \right).$$

For $0 \leq k \leq m$, the function $G_{m,k}(t)$ is a polynomial in t of degree $m - k$. The coefficients A_k are defined by

$$A_k = \sum_{j=k}^m \mathbf{S}_j^{(k)} \sigma_{m-j}, \quad A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}, \quad A_m = 1, \quad (1.7)$$

where $\mathbf{S}_j^{(k)}$ is the Stirling number of the second kind and the coefficients σ_j ($0 \leq j \leq m$) are generated by

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^m \sigma_{m-j} x^j.$$

The case $m_1 = \cdots = m_r = 1$ in (1.5) has been given earlier in [8], and also in [6] using different methods. When (m_r) is empty we define $m = 0$; in this case (1.5) reduces to the Euler transformation (1.3). In Section 2, we shall employ the same approach described in [14, p. 49] to the transformation formula (1.5) in our proof of the extension of Saalschütz's theorem to ${}_{r+3}F_{r+2}$ series.

2. The extension of Saalschütz's theorem

The extension of Saalschütz's summation theorem is given by the following:

Theorem 1. *Let (m_r) be a set of positive integers and define $m \equiv m_1 + \cdots + m_r$. Let n be a nonnegative integer. Then*

$${}_{r+3}F_{r+2} \left(\begin{matrix} -n, a, b, & (f_r + m_r) \\ c, 1 + a + b - c + m - n, & (f_r) \end{matrix} \middle| 1 \right) = \frac{(c - a - m)_n (c - b - m)_n ((\eta_m + 1))_n}{(c)_n (c - a - b - m)_n ((\eta_m))_n}, \quad (2.1)$$

where (η_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m defined in (1.6).

Proof: From (1.5) we have

$$\begin{aligned} & {}_{m+2}F_{m+1} \left(\begin{matrix} c - a - m, c - b - m, (\eta_m + 1) \\ c, (\eta_m) \end{matrix} \middle| x \right) \\ &= (1 - x)^{m+a+b-c} {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| x \right). \end{aligned}$$

The coefficient of x^n on the left-hand side is

$$\frac{(c - a - m)_n (c - b - m)_n ((\eta_m + 1))_n}{(c)_n n! ((\eta_m))_n} \quad (2.2)$$

and this must equal the coefficient of x^n on the right-hand side. With

$$D_k \equiv (-1)^k \binom{m + a + b - c}{k},$$

this latter coefficient is given by

$$\begin{aligned}
& \sum_{k=0}^n D_{n-k} \frac{(a)_k (b)_k ((f_r + m_r))_k}{(c)_k k! ((f_r))_k} \\
&= \sum_{k=0}^n \frac{(-1)^{n-k} \Gamma(1+a+b-c+m)}{(n-k)! \Gamma(1+a+b-c+m-n+k)} \frac{(a)_k (b)_k ((f_r + m_r))_k}{(c)_k k! ((f_r))_k} \\
&= \frac{(c-a-b-m)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{k!} \frac{(a)_k (b)_k ((f_r + m_r))_k}{(c)_k (1+a+b-c+m-n)_k ((f_r))_k} \\
&= \frac{(c-a-b-m)_n}{n!} {}_{r+3}F_{r+2} \left(\begin{matrix} -n, a, b, & (f_r + m_r) \\ c, 1+a+b-c+m-n, & (f_r) \end{matrix} \middle| 1 \right),
\end{aligned}$$

where we have used the identities

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, \quad \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha-n)} = (-1)^n (\alpha)_n.$$

Equating this coefficient to that in (2.2) we then obtain the desired summation (2.1) \square

3. Examples

In the case $r = 1$ and $m_1 = m = 1$ we have from (2.1)

$${}_4F_3 \left(\begin{matrix} -n, a, b, & f+1 \\ c, 2+a+b-c-n, & f \end{matrix} \middle| 1 \right) = \frac{(c-a-1)_n (c-b-1)_n (\eta+1)_n}{(c)_n (c-a-b-1)_n (\eta)_n}, \quad (3.1)$$

where η is the zero of the first-degree parametric polynomial obtained from (1.6)

$$Q_1(t) = -\{f(c-a-b-1) + ab\}t + (c-a-1)(c-b-1)f,$$

whence

$$\eta = \frac{(c-a-1)(c-b-1)f}{ab + (c-a-b-1)f}.$$

This result was derived in [13] using a different approach.

A special case of (3.1), which may be of some interest, is obtained when $c = 1 + a - b$, so that $\eta = (a - 2b)f / (2f - a)$, to yield

$${}_4F_3 \left(\begin{matrix} -n, a, b, & f+1 \\ 1+a-b, 1+2b-n, & f \end{matrix} \middle| 1 \right) = \frac{(a-2b)_n (-b)_n}{(1+a-b)_n (-2b)_n} \left\{ 1 + \frac{n(2f-a)}{(a-2b)f} \right\},$$

where we have used the fact that $(\eta+1)_n / (\eta)_n = (\eta+n)/\eta$. This result was obtained earlier by Kim, Rathie and Paris by other means who used it to obtain a reduction formula for the Kampé de Fériet function [2]. In the above result, if we let $f = \frac{1}{2}a$ we obtain

$${}_4F_3 \left(\begin{matrix} -n, a, b, 1 + \frac{1}{2}a \\ c, 1+2b-n, \frac{1}{2}a \end{matrix} \middle| 1 \right) = \frac{(a-2b)_n (-b)_n}{(1+a-b)_n (-2b)_n} \quad (a \neq 2b),$$

which is a known summation; see [14, Appendix III, Eq. (17)].

Also, in the case $r = 1$ with $m = 2$, we find from (1.7) that $A_0 = f(1 + f)$, $A_1 = 2(1 + f)$ and $A_2 = 1$. Introducing the abbreviated notation

$$\lambda \equiv c - a - 2, \quad \lambda' \equiv c - b - 2, \quad \sigma \equiv c - a - b - 2,$$

we obtain from (1.6) the quadratic parametric polynomial (with zeros η_1 and η_2) given by

$$\begin{aligned} Q_2(t) &= A_0(\lambda)_2(\lambda')_2 \left\{ 1 - \frac{2\sigma t}{\lambda\lambda'} + \frac{(t)_2(\sigma)_2}{(\lambda)_2(\lambda')_2} \right\} - A_1abt(\lambda + 1)(\lambda' + 1) \left\{ 1 - \frac{\sigma(1+t)}{(\lambda + 1)(\lambda' + 1)} \right\} \\ &\quad + A_2(a)_2(b)_2t(1+t) \\ &= A_0(\lambda)_2(\lambda')_2 \left\{ 1 - \frac{2Bt}{\lambda\lambda'} + \frac{Ct(1+t)}{(\lambda)_2(\lambda')_2} \right\}, \end{aligned}$$

where

$$B \equiv \sigma + \frac{ab}{f}, \quad C \equiv (\sigma)_2 + \frac{2ab\sigma}{f} + \frac{(a)_2(b)_2}{(f)_2}.$$

Then, from (2.1),

$${}_4F_3 \left(\begin{matrix} -n, a, b, f+2 \\ c, 3+a+b-c-n, f \end{matrix} \middle| 1 \right) = \frac{(c-a-2)_n(c-b-2)_n}{(c)_n(c-a-b-2)_n} \frac{(\eta_1+1)_n(\eta_2+1)_n}{(\eta_1)_n(\eta_2)_n}. \quad (3.2)$$

For example, if $a = \frac{3}{2}$, $b = 2$, $c = \frac{5}{4}$ and $f = 1$ we find

$$Q_2(t) = \frac{3465}{256} \left\{ 1 + \frac{40}{99}t - \frac{16}{495}t(1+t) \right\},$$

which has the zeros $\eta_1 = -\frac{9}{4}$ and $\eta_2 = \frac{55}{4}$.

Finally, we consider the case $r = 2$ with $m_1 = m_2 = 1$ (so that $m = 2$). If we take $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = 1$, $f_1 = \frac{1}{4}$ and $f_2 = 2$, we find the parametric polynomial

$$Q_2(t) = \frac{1}{72} \left\{ 12 + \frac{31}{2}t + \frac{15}{2}t(1+t) \right\}$$

which has the zeros $\eta_1 = -\frac{2}{3}$ and $\eta_2 = -\frac{12}{5}$. Then from (2.1) we obtain

$${}_5F_4 \left(\begin{matrix} -n, a, b, \frac{5}{4}, 3 \\ c, 3+a+b-c-n, \frac{1}{4}, 2 \end{matrix} \middle| 1 \right) = \frac{(c-a-2)_n(c-b-2)_n}{(c)_n(c-a-b-2)_n} \frac{(\frac{1}{3})_n(-\frac{7}{5})_n}{(-\frac{2}{3})_n(-\frac{12}{5})_n}.$$

Similarly, other results can also be obtained.

Remark. Two other quadratic transformations, different from that in (1.4), have been presented by Miller and Paris [9, Section 6] in the form

$$(1+x)^{-2a} {}_3F_2 \left(\begin{matrix} a, a + \frac{1}{2}, f+1 \\ c, f \end{matrix} \middle| \frac{x^2}{(1+x)^2} \right) = {}_4F_3 \left(\begin{matrix} 2a, c - \frac{3}{2}, \xi_1 + 1, \xi_2 + 1 \\ 2c - 1, \xi_1, \xi_2 \end{matrix} \middle| -2x \right) \quad (3.3)$$

provided $c \neq \frac{3}{2}$, where

$$\xi_{1,2} = 2f - \frac{1}{2} \pm [(2f - \frac{1}{2})^2 - 4(c - \frac{3}{2})f]^{1/2},$$

and

$$(1+x)^{-2a} {}_3F_2 \left(\begin{matrix} a, a + \frac{1}{2}, f+1 \\ c, f \end{matrix} \middle| \frac{4x}{(1+x)^2} \right) = {}_4F_3 \left(\begin{matrix} 2a, 2a - c + 1, \eta_1 + 1, \eta_2 + 1 \\ c, \eta_1, \eta_2 \end{matrix} \middle| x \right) \quad (3.4)$$

provided $c \neq 2a + 1$, where

$$\eta_{1,2} = a \pm [a^2 - (2a - c + 1)f]^{1/2}.$$

In the same paper, the authors also derived the extension of both these quadratic transformations in the more general case of r pairs of numeratorial and denominatorial parameters differing by positive integers. It should be pointed out that the results in (3.3) and (3.4) were also obtained in [11] and [12], respectively, following different methods.

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