Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function

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INEQUALITIES, ASYMPTOTIC EXPANSIONS AND COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTION

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Abstract. In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma\left(x+\frac{1}{2}\right)$.

1. Introduction

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \qquad (x \in I;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.1}$$

Dubourdieu [13, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (1.1). See also [16] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [48, p. 161]). The main properties of completely monotonic functions are given in [48, Chapter IV]. We also refer to [4], where a extensive list of references on completely monotonic functions can be found.

Euler's gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function. The derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\psi(x)\}, \qquad n \in \mathbb{N}$$

are called the polygamma functions.

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In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma\left(x+\frac{1}{2}\right)$.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Lemmas

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by the generating functions

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

The rational numbers $B_n=B_n(0)$ and integers $E_n=2^nE_n(1/2)$ are called Bernoulli and Euler numbers, respectively.

It follows from Problem 154 in Part I, Chapter 4, of [39] that

$$\sum_{j=1}^{2m}\frac{B_{2j}}{(2j)!}t^{2j}<\frac{t}{e^t-1}-1+\frac{t}{2}<\sum_{j=1}^{2m+1}\frac{B_{2j}}{(2j)!}t^{2j} \tag{2.1}$$

for $t>0$ and $m\in\mathbb{N}_0$. The inequality (2.1) can be also found in [17, 40].

Lemma 1 presents an analogous result to (2.1).

Lemma 1. *For $x>0$ and $m\in\mathbb{N}$,*

$$\sum_{j=2}^{2m+1}\frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!}<\frac{2}{e^x+1}-1+\frac{x}{2}<\sum_{j=2}^{2m}\frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!}, \tag{2.2}$$

where B_n $(n\in\mathbb{N}_0)$ are the Bernoulli numbers.

Proof. The noted Boole's summation formula (see [45, p. 17]) states for $k\in\mathbb{N}$ that

$$f(1)=\frac{1}{2}\sum_{j=0}^{k-1}\frac{E_j(1)}{j!}\left(f^{(j)}(1)+f^{(j)}(0)\right)+\frac{1}{2(k-1)!}\int_0^1 f^{(k)}(t)E_{k-1}(t)\mathrm{d}t,$$

which can be written for $m\in\mathbb{N}$ as

$$f(1)-f(0)=\sum_{j=1}^{m}\frac{E_{2j-1}(1)}{(2j-1)!}\left(f^{(2j-1)}(1)+f^{(2j-1)}(0)\right)+\frac{1}{(2m-1)!}\int_0^1 f^{(2m)}(t)E_{2m-1}(t)\mathrm{d}t. \tag{2.3}$$

Applying formula (2.3) to $f(t)=e^{xt}$, we obtain

$$-\frac{2}{e^x+1}+1-\frac{x}{2}=\sum_{j=2}^{m}\frac{E_{2j-1}(1)}{(2j-1)!}x^{2j-1}+\frac{x}{e^x+1}\frac{x^{2m-1}}{(2m-1)!}\int_0^1 e^{xt}E_{2m-1}(t)\mathrm{d}t. \tag{2.4}$$

It is well known (see [1, p. 804]) that

$$E_{2m+1}(1-t)=-E_{2m+1}(t) \quad \text{and} \quad E_{2m+1}(\tfrac{1}{2})=0.$$

Noting that

$$E_{4m-1}(t)>0, \quad E_{4m+1}(t)<0 \quad \text{for} \quad 0<t<1/2, \quad m=1,2,\ldots,$$

we imply for $x>0$ that

$$\int_0^1 e^{xt}E_{4m-1}(t)\mathrm{d}t=\int_0^{1/2}\left(e^{xt}-e^{x(1-t)}\right)E_{4m-1}(t)\mathrm{d}t<0$$

and

$$\int_0^1 e^{xt}E_{4m+1}(t)\mathrm{d}t=\int_0^{1/2}\left(e^{xt}-e^{x(1-t)}\right)E_{4m+1}(t)\mathrm{d}t>0.$$

Combining these with (2.4), we immediately obtain that for $x>0$ and $m\in\mathbb{N}$,

$$-\sum_{j=2}^{2m+1}\frac{E_{2j-1}(1)}{(2j-1)!}x^{2j-1}<\frac{2}{e^x+1}-1+\frac{x}{2}<-\sum_{j=2}^{2m}\frac{E_{2j-1}(1)}{(2j-1)!}x^{2j-1}. \tag{2.5}$$

Noting that

$$E_n(1)=\frac{2(2^{n+1}-1)}{n+1}B_{n+1},\qquad n\in\mathbb{N},$$

the inequality (2.5) can be written as (2.2). The proof of Theorem 1 is complete. □

The inequality (2.2) can be written for $x>0$ and $m\in\mathbb{N}_0$ as

$$\sum_{j=1}^{2m+1}\frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!}<\frac{2}{e^x+1}-1<\sum_{j=1}^{2m}\frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!}, \tag{2.6}$$

i.e.,

$$(-1)^{m+1}\left(\frac{2}{e^x+1}-1-\sum_{j=1}^{m}\frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!}\right)>0. \tag{2.7}$$

Lemma 2 ([10])**.** *Let $r\neq 0$ be a given real number and $\ell\geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}\right)^{x^\ell/r},\qquad x\to\infty, \tag{2.8}$$

where the coefficients $b_j\equiv b_j(\ell,r)$ $(j\in\mathbb{N})$ are given by

$$b_j\equiv b_j(\ell,r)=\sum\frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{B_2}{1\cdot 2}\right)^{k_1}\left(\frac{B_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{B_{j+1}}{j(j+1)}\right)^{k_j}, \tag{2.9}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j.$$

Lemma 3. *Let $r\neq 0$ be a given real number and $\ell\geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x+\frac{1}{2}\right)\sim\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(1+\sum_{j=1}^{\infty}\frac{c_j}{x^j}\right)^{x^\ell/r},\qquad x\to\infty, \tag{2.10}$$

where the coefficients $c_j \equiv c_j(\ell, r)\,(j \in \mathbb{N})$ *are given by*

$$c_j = \sum \frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(1-2^{-1})B_2}{1\cdot 2}\right)^{k_1}\left(\frac{(1-2^{-3})B_4}{3\cdot 4}\right)^{k_2}\cdots\left(\frac{(1-2^{1-2j})B_{2j}}{(2j-1)(2j)}\right)^{k_j}, \tag{2.11}$$

summed over all nonnegative integers k_j *satisfying the equation*

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

Proof. The following asymptotic expansion can be found [27, p. 32]

$$\ln\Gamma\left(x+\frac{1}{2}\right) \sim x\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty}\frac{B_{2j}(\frac{1}{2})}{2j(2j-1)}x^{1-2j}, \qquad x\to\infty. \tag{2.12}$$

It is well-known (see [1, p. 805]) that

$$B_n(\tfrac{1}{2}) = -(1-2^{1-n})B_n, \qquad n \in \mathbb{N}_0,$$

and then the expansion (2.12) can be rewritten as

$$\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x} = \exp\left(\sum_{k=1}^{m}\frac{-(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k-1}} + \mathcal{R}_m(x)\right), \qquad x\to\infty, \tag{2.13}$$

where $\mathcal{R}_m(x) = O(1/x^{2m+1})$. Further, we have

$$\begin{aligned}
\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x}\right)^{r/x^\ell} &= e^{r\mathcal{R}_m(x)/x^\ell}\exp\left(\sum_{k=1}^{m}\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}}\right)\\
&= e^{r\mathcal{R}_m(x)/x^\ell}\prod_{k=1}^{m}\left[1+\left(\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}}\right)+\frac{1}{2!}\left(\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}}\right)^2+\cdots\right]\\
&= e^{r\mathcal{R}_m(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}\\
&\quad\times\left(\frac{-r(1-2^{-1})B_2}{1\cdot 2}\right)^{k_1}\left(\frac{-r(1-2^{-3})B_4}{3\cdot 4}\right)^{k_2}\cdots\left(\frac{-r(1-2^{1-2m})B_{2m}}{(2m-1)(2m)}\right)^{k_m}\\
&\quad\times\frac{1}{x^{(1+\ell)k_1+(3+\ell)k_2+\cdots+(2m+\ell-1)k_m}}.
\end{aligned} \tag{2.14}$$

On the other hand, from (2.13) it follows that for any positive integer m,

$$\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x}\right)^{r/x^\ell} = 1 + \sum_{j=1}^{m}\frac{c_j}{x^j} + O(1/x^{m+1}) \tag{2.15}$$

for some real numbers $c_1, \ldots, c_m$.

Equating the coefficients by the equal powers of x in (2.14) and (2.15), we see that

$$c_j = \sum \frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(1-2^{-1})B_2}{1\cdot 2}\right)^{k_1}\left(\frac{(1-2^{-3})B_4}{3\cdot 4}\right)^{k_2}\cdots\left(\frac{(1-2^{1-2j})B_{2j}}{(2j-1)(2j)}\right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

This completes the proof of Lemma 3. □

Lemma 4. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(1+\sum_{j=1}^{\infty}\frac{p_j}{x^j}\right)^{x^{\ell}/r}, \qquad x \to \infty, \tag{2.16}$$

where the coefficients $p_j \equiv p_j(\ell, r)$ $(j \in \mathbb{N})$ are given by

$$p_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1}\left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2}\cdots\left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j}, \tag{2.17}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

Proof. From (3.7) we obtain the following asymptotic expansion:

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x}\exp\left(\sum_{j=1}^{\infty}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right), \qquad x \to \infty. \tag{2.18}$$

Write (2.18) as

$$\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})} \sim \exp\left(\sum_{k=1}^{m}\frac{(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k-1}} + R_m(x)\right), \qquad x \to \infty, \tag{2.19}$$

where $R_m(x) = O(1/x^{2m+1})$. Further, we have

$$\begin{aligned}
\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)^{r/x^{\ell}} &= e^{rR_m(x)/x^{\ell}}\exp\left(\sum_{k=1}^{m}\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}}\right) \\
&= e^{rR_m(x)/x^{\ell}}\prod_{k=1}^{m}\left[1+\left(\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}}\right)+\frac{1}{2!}\left(\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}}\right)^2+\cdots\right] \\
&= e^{rR_m(x)/x^{\ell}}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!} \\
&\quad\times\left(\frac{r(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1}\left(\frac{r(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2}\cdots\left(\frac{r(2^{2m}-1)B_{2m}}{m(2m-1)2^{2m}}\right)^{k_m} \\
&\quad\times\frac{1}{x^{(1+\ell)k_1+(3+\ell)k_2+\cdots+(2m+\ell-1)k_m}}.
\end{aligned} \tag{2.20}$$

On the other hand, from (2.19) it follows that for any positive integer m,

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)^{r/x^{\ell}} = 1+\sum_{j=1}^{m}\frac{p_j}{x^j} + O(1/x^{m+1}) \tag{2.21}$$

for some real numbers $p_1, \ldots, p_m$.

Equating the coefficients by the equal powers of x in (2.20) and (2.21), we see that

$$p_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1}\left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2}\cdots\left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

This completes the proof of Lemma 4. ☐

Lemma 5. *For $t \neq 0$,*

$$\frac{1}{2t^2}+\frac{1}{12}-\frac{7}{240}t^2+\frac{31}{6048}t^4-\frac{127}{172800}t^6<\frac{\cosh t}{\cosh(2t)-1}<\frac{1}{2t^2}+\frac{1}{12}-\frac{7}{240}t^2+\frac{31}{6048}t^4. \tag{2.22}$$

Proof. We only prove the second inequality in (2.22). The proof of the first inequality in (2.22) is analogous. By using the power series expansion of $\cosh t$, we find that

$$\left(\frac{1}{2t^2}+\frac{1}{12}-\frac{7}{240}t^2+\frac{31}{6048}t^4\right)\Big(\cosh(2t)-1\Big)-\cosh t=\sum_{n=4}^{\infty}\frac{\alpha_n}{(2n+2)!}t^{2n}$$

with

$$\begin{aligned}\alpha_n&=2^{2n-4}\left(\frac{104}{3}+\frac{10513}{1260}n+\frac{11117}{1890}n^2-\frac{1577}{1260}n^3-\frac{4303}{1890}n^4-\frac{31}{63}n^5+\frac{62}{189}n^6\right)\\&\quad-2(n+1)(2n+1)\\&=2^{2n-4}\bigg(339+\frac{1005649}{1260}(n-4)+\frac{1355243}{1890}(n-4)^2+\frac{382391}{1260}(n-4)^3\\&\quad+\frac{125897}{1890}(n-4)^4+\frac{155}{21}(n-4)^5+\frac{62}{189}(n-4)^6\bigg)-2(n+1)(2n+1)\\&>2^{2n-4}\cdot 339-2(n+1)(2n+1)\\&>0\quad\text{for}\quad n\geq 4.\end{aligned}$$

Hence, the second inequality in (2.22) holds. This completes the proof of Lemma 5. ☐

3. Completely monotonic functions

It is known in [45, p. 64] that

$$\frac{t}{e^t-1}-1+\frac{t}{2}=\sum_{j=1}^{n}\frac{B_{2j}}{(2j)!}t^{2j}+(-1)^n t^{2n+2}\nu_n(t),\qquad n\geq 0, \tag{3.1}$$

where

$$\nu_n(t)=\sum_{k=1}^{\infty}\frac{2}{(t^2+4\pi^2k^2)(2\pi k)^{2n}}>0.$$

It is easy to see that (3.1) implies (2.1).

The noted Binet's first formula [44, p. 16] states that

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}+\int_0^{\infty}\left(\frac{t}{e^t-1}-1+\frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t,\qquad x>0. \tag{3.2}$$

Combining (3.1) with (3.2), Xu and Han [46] educed in 2009 that for every $m\in\mathbb{N}_0$, the function

$$R_m(x)=(-1)^m\left[\ln\Gamma(x)-\left(x-\frac{1}{2}\right)\ln x+x-\ln\sqrt{2\pi}-\sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right] \tag{3.3}$$

is completely monotonic on $(0,\infty)$.

For $m=0$, complete monotonicity property of $R_m(x)$ was proved by Muldoon [38]. Alzer [2] first proved in 1997 that $R_m(x)$ is completely monotonic on $(0,\infty)$. In 2006, Koumandos [17] proved double inequality (2.1), and then used (2.1) and (3.2) to give the proof of complete monotonicity property of $R_m(x)$. In 2009, Koumandos and Pedersen [18, Theorem 2.1] strengthened this result.

Based on the inequality (2.2), in this section we prove that for every $m \in \mathbb{N}_0$, the function

$$F_m(x) = (-1)^m \left[\ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) - \frac{1}{2}\ln x - \sum_{j=1}^{m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}} \right] \tag{3.4}$$

is completely monotonic on $(0,\infty)$. This result is similar to complete monotonicity property of $R_m(x)$ in (3.3).

Theorem 1. *For every $m \in \mathbb{N}_0$, the function $F_m(x)$, defined by* (3.4)*, is completely monotonic on* $(0,\infty)$.

Proof. The logarithm of the gamma function has the following integral representation (see [1, p. 258]):

$$\ln\Gamma(x) = \int_0^\infty \left[(x-1)e^{-t} - \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\right]\frac{\mathrm{d}t}{t}, \qquad x>0. \tag{3.5}$$

By using (3.5) and the following representations:

$$\ln x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}\mathrm{d}t, \qquad x>0$$

in [1, p. 230, 5.1.32] and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\mathrm{d}t, \quad x>0 \quad \text{and} \quad r>0$$

in [1, p. 255, 6.1.1], we find that

$$\begin{aligned} F_m(x) &= (-1)^m \left[\int_0^\infty \left(-\frac{1}{e^{t/2}+1}+\frac{1}{2}\right)\frac{e^{-xt}}{t}\mathrm{d}t \right. \\ &\qquad \left. - \sum_{j=1}^{m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)!}\int_0^\infty t^{2j-2}e^{-xt}\mathrm{d}t \right] \\ &= \frac{1}{2}\int_0^\infty (-1)^{m+1}\lambda_m(t)\frac{e^{-xt}}{t}\,\mathrm{d}t, \end{aligned} \tag{3.6}$$

where

$$\lambda_m(t) = \frac{2}{e^{t/2}+1} - 1 - \sum_{j=1}^{m}\frac{(1-2^{2j})B_{2j}}{j(2j-1)!}\left(\frac{t}{2}\right)^{2j-1}.$$

By (2.7), we have $(-1)^{m+1}\lambda_m(t) > 0$ for $t>0$ and $m \in \mathbb{N}_0$. From (3.6) we obtain that for every $m \in \mathbb{N}_0$,

$$(-1)^n F_m^{(n)}(x) = \frac{1}{2}\int_0^\infty (-1)^{m+1}\lambda_m(t)t^{n-1}e^{-xt}\,\mathrm{d}t > 0$$

for $x>0$ and $n \in \mathbb{N}_0$. The proof of Theorem 1 is complete. □

Under the inequality $(-1)^n F_m^{(n)}(x) > 0$ for $x>0$ and $m,n \in \mathbb{N}_0$, we obtain the following

Corollary 1. (i) *Let $m \in \mathbb{N}_0$. Then for $x > 0$,*

$$\sqrt{x}\exp\left(\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x}\exp\left(\sum_{j=1}^{2m+1}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right). \tag{3.7}$$

(ii) *Let $m, n \in \mathbb{N}$. Then for $x > 0$,*

$$\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}} < (-1)^n\left(\psi^{(n-1)}(x+1)-\psi^{(n-1)}\left(x+\frac{1}{2}\right)\right)+\frac{(n-1)!}{2x^n} < \sum_{j=1}^{2m-1}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}}. \tag{3.8}$$

By using the obtained results above, we here present inequalities and integral representations for the constant π.

The problem of finding new and sharp inequalities for the gamma function Γ and in particular about the Wallis ratio

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} = \frac{1}{\sqrt{\pi}}\cdot\frac{(2n)!!}{(2n-1)!!}, \qquad n \in \mathbb{N} \tag{3.9}$$

has attracted the attention of many researchers (see [11, 19, 20, 33] and references therein). Some inequalities for π can be found (see, for example, [15, 21, 34, 35]). Here, we employ the special double factorial notation as follows:

$$(2n)!! = 2\cdot 4\cdot 6\cdots(2n) = 2^n n!,$$
$$(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1) = \pi^{-1/2}2^n\Gamma(n+\tfrac{1}{2}),$$
$$0!! = 1, \qquad (-1)!! = 1$$

(see [1, p. 258]). Very recently, Lin [21, Theorem 2.4] proved that for all $n \in \mathbb{N}$,

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\frac{1}{4n}+\frac{1}{96n^3}-\frac{1}{320n^5}+\frac{17}{7168n^7}-\frac{31}{9216n^9}\right) < \pi < \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\frac{1}{4n}+\frac{1}{96n^3}-\frac{1}{320n^5}+\frac{17}{7168n^7}\right). \tag{3.10}$$

Setting $x = n$ in (3.7), we obtain estimate for the constant π:

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\sum_{j=1}^{2m+1}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{j(2j-1)n^{2j-1}}\right) < \pi < \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{j(2j-1)n^{2j-1}}\right). \tag{3.11}$$

Obviously, (3.11) is a generalization of (3.10).

Formula (3.6) gives the following integral representation:

$$\begin{aligned}
&\ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)-\frac{1}{2}\ln x-\sum_{j=1}^{m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\\
&=-\frac{1}{2}\int_0^\infty\left(\frac{2}{e^{t/2}+1}-1-\sum_{j=1}^{m}\frac{(1-2^{2j})B_{2j}}{j(2j-1)!}\left(\frac{t}{2}\right)^{2j-1}\right)\frac{e^{-xt}}{t}\,\mathrm{d}t, \qquad (3.12)
\end{aligned}$$

which implies

$$\begin{aligned}
&\psi(x+1)-\psi\left(x+\frac{1}{2}\right)-\frac{1}{2x}+\sum_{j=1}^{m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{jx^{2j}}\\
&=\frac{1}{2}\int_0^\infty\left(\frac{2}{e^{t/2}+1}-1-\sum_{j=1}^{m}\frac{(1-2^{2j})B_{2j}}{j(2j-1)!}\left(\frac{t}{2}\right)^{2j-1}\right)e^{-xt}\,\mathrm{d}t \qquad (3.13)
\end{aligned}$$

for $x>0$ and $m\in\mathbb{N}_0$. Formulas (3.12) and (3.13) can provide integral representations for the constant π. For example, setting $(x,m)=(1/2,0)$ in (3.12) yields

$$\int_0^\infty\left(1-\frac{2}{e^u+1}\right)\frac{e^{-u}}{u}\,\mathrm{d}u=\ln\left(\frac{\pi}{2}\right). \qquad (3.14)$$

Setting $(x,m)=(1/4,1)$ in (3.13) yields

$$\int_0^\infty\left(\frac{2}{e^u+1}-1+\frac{u}{2}\right)e^{-u/2}\,\mathrm{d}u=4-\pi. \qquad (3.15)$$

Many formulas exist for the representation of π, and a collection of these formulas is listed [42, 43]. For more history of π see [6, 7, 14].

Very recently, Mortici et al. [36] proved some completely monotonic functions and inequalities associated with the ratio of gamma functions.

4. Asymptotic expansions

Stirling's formula

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n\in\mathbb{N}:=\{1,2,\dots\} \qquad (4.1)$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n!\sim\text{constant}\cdot\sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\begin{aligned}
\Gamma(x+1)&\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\exp\left(\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \qquad (4.2)\\
&=\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\cdots\right)
\end{aligned}$$

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers. The following asymptotic formula is due to Laplace:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right) \tag{4.3}$$

as $x \to \infty$ (see [1, p. 257, Eq. (6.1.37)]). The expression (4.3) is sometimes incorrectly called Stirling's series (see [12, pp. 2-3]). Stirling's formula is in fact the first approximation to the asymptotic formula (4.3). Stirling's formula has attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements (see, for example, [8, 9, 22, 23, 24, 25, 26, 29, 30, 31, 32, 37] and the references cited therein). See also an overview at [28].

Windschitl (see [5, p. 128] and [49]) had noted that for $x > 8$, the approximation

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} \tag{4.4}$$

gives at least eight decimal places of the gamma function. The formula (4.4) derives

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty. \tag{4.5}$$

Inspired by (4.4) and (4.5), Alzer [3] proved in 2009 that for all $x > 0$,

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + \frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + \frac{\beta}{x^5}\right) \tag{4.6}$$

with the best possible constants $\alpha = 0$ and $\beta = 1/1620$.

In 2014, Lu *et al.* [24] extended Windschitl's formula as follows:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh\left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \frac{a_{11}}{n^{11}} + \cdots\right)\right)^{n/2}, \tag{4.7}$$

where

$$a_7 = \frac{1}{810}, \quad a_9 = -\frac{67}{42\,525}, \quad a_{11} = \frac{19}{8505}, \ldots. \tag{4.8}$$

However, the authors did not give the general formula for the coefficients a_j ($j \geq 7$) in (4.7). Subsequently, Chen [9] gave a recurrence relation formula for determining the coefficient of n^{-j} ($j \in \mathbb{N}$) in (4.7). Also in [9], Chen developed Windschitl's approximation formula to a new asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2 + \sum_{j=0}^{\infty} r_j x^{-j}}, \qquad x \to \infty, \tag{4.9}$$

and provided a recurrence relation for determining the coefficients r_j in (4.9).

Smith [41, Eq. (43)] presented the following analogous result to (4.5):

$$\Gamma\left(x + \frac{1}{2}\right) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x}\right)^{x/2} \left(1 + O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty. \tag{4.10}$$

Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. We here determine the coefficients $a_j \equiv a_j(\ell, r)$ and $d_j \equiv d_j(\ell, r)$ (for $j \in \mathbb{N}$) such that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right)^{x^\ell/r}$$

and

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x} + \sum_{j=1}^{\infty} \frac{d_j}{x^j}\right)^{x^\ell/r}$$

as $x \to \infty$.

Theorem 2. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{4.11}$$

with the coefficients $a_j \equiv a_j(\ell, r)$ $(j \in \mathbb{N})$ given by

$$a_{2j-1} = b_{2j-1}(\ell, r), \quad a_{2j} = b_{2j}(\ell, r) - \frac{1}{(2j+1)!}, \tag{4.12}$$

where $b_j(\ell, r)$ $(j \in \mathbb{N})$ can be calculated using (2.9).

Proof. The Maclaurin expansion of $\sinh t$ with $t = 1/x$ gives

$$x \sinh\frac{1}{x} = 1 + \sum_{j=1}^{\infty} \frac{1}{(2j+1)!x^{2j}}, \qquad x \neq 0. \tag{4.13}$$

In view of (2.8) and (4.13), we can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^{r/x^\ell} - x \sinh\frac{1}{x} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty, \tag{4.14}$$

where a_j $(j \in \mathbb{N})$ are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty} \frac{b_j}{x^j} - \sum_{j=1}^{\infty} \frac{1}{(2j+1)!x^{2j}} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty, \tag{4.15}$$

where $b_j \equiv b_j(\ell, r)$ $(j \in \mathbb{N})$ are given in (2.9). Equating coefficients of equal powers of x in (4.15) yields (4.12). The proof of Theorem 2 is complete. □

Remark 1. *Setting $(r, \ell) = (2, 1)$ in* (4.11) *yields full asymptotic expansion of Windschitl's formula* (4.4)*:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \frac{1}{810x^6} - \frac{163}{170100x^8} + \frac{1019}{680400x^{10}} - \cdots\right)^{x/2} \tag{4.16}$$

as $x \to \infty$.

Theorem 3. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x} + \sum_{j=1}^{\infty}\frac{d_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{4.17}$$

with the coefficients $d_j \equiv d_j(\ell, r)\,(j \in \mathbb{N})$ given by

$$d_{2j-1} = c_{2j-1}(\ell, r), \quad d_{2j} = c_{2j}(\ell, r) - \frac{4(2^{2j+2}-1)B_{2j+2}}{(2j+2)!}, \tag{4.18}$$

where $c_j(\ell, r)\,(j \in \mathbb{N})$ can be calculated using (2.11).

Proof. The power series expansion of $\tanh t$ with $t = 1/(2x)$ gives

$$2x\tanh\frac{1}{2x} = 1 + \sum_{j=2}^{\infty}\frac{4(2^{2j}-1)B_{2j}}{(2j)!}\frac{1}{x^{2j-2}}, \qquad |x| > \frac{1}{\pi}. \tag{4.19}$$

In view of (2.10) and (4.19), we can let

$$\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}(x/e)^x}\right)^{r/x^\ell} - 2x\tanh\frac{1}{2x} \sim \sum_{j=1}^{\infty}\frac{d_j}{x^j}, \qquad x \to \infty, \tag{4.20}$$

where d_j $(j \in \mathbb{N})$ are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty}\frac{c_j}{x^j} - \sum_{j=1}^{\infty}\frac{4(2^{2j+2}-1)B_{2j+2}}{(2j+2)!}\frac{1}{x^{2j}} \sim \sum_{j=1}^{\infty}\frac{d_j}{x^j}, \qquad x \to \infty, \tag{4.21}$$

where $c_j \equiv c_j(\ell, r)\,(j \in \mathbb{N})$ are given in (2.11). Equating coefficients of equal powers of x in (4.21) yields (4.18). The proof of Theorem 3 is complete. □

Remark 2. *Setting $(r, \ell) = (2, 1)$ in (4.17) yields the following full asymptotic expansion:*

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x} - \frac{31}{25920x^6} + \frac{6829}{5443200x^8} - \cdots\right)^{x/2} \tag{4.22}$$

as $x \to \infty$.

From (4.5) and (4.10), we derive

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \left(1 + O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty.$$

This fact motivated us to observe the following

Theorem 4. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x} + \sum_{j=1}^{\infty}\frac{q_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{4.23}$$

with the coefficients $q_j \equiv q_j(\ell, r)\,(j \in \mathbb{N})$ given by

$$q_{2j-1} = p_{2j-1}(\ell, r), \quad q_{2j} = p_{2j}(\ell, r) - \frac{1}{2^{2j}(2j)!}, \tag{4.24}$$

where $p_j(\ell, r)\,(j \in \mathbb{N})$ can be calculated using (2.17).

Proof. The Maclaurin expansion of $\cosh t$ with $t = 1/(2x)$ gives

$$\cosh\frac{1}{2x} = 1 + \sum_{j=1}^{\infty} \frac{1}{2^{2j}(2j)!}\frac{1}{x^{2j}}, \qquad |x| \neq 0. \tag{4.25}$$

In view of (2.16) and (4.25), we can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)^{r/x^{\ell}} - \cosh\frac{1}{2x} \sim \sum_{j=1}^{\infty}\frac{q_j}{x^j}, \qquad x\to\infty, \tag{4.26}$$

where q_j ($j \in \mathbb{N}$) are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty}\frac{p_j}{x^j} - \sum_{j=1}^{\infty} \frac{1}{2^{2j}(2j)!}\frac{1}{x^{2j}} \sim \sum_{j=1}^{\infty}\frac{q_j}{x^j}, \qquad x\to\infty, \tag{4.27}$$

where $p_j \equiv p_j(\ell, r)$ ($j \in \mathbb{N}$) are given in (2.17). Equating coefficients of equal powers of x in (4.27) yields (4.24). The proof of Theorem 3 is complete. □

Remark 3. *Setting $(r,\ell) = (1,1)$ in (4.23) yields the following full asymptotic expansion:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x} + \frac{7}{5760x^6} - \frac{65}{64512x^8} + \cdots\right)^x, \qquad x\to\infty. \tag{4.28}$$

5. Inequalities

Theorem 5. *For $x > 0$,*

$$\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\left(1+\frac{\theta_1}{x^5}\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\left(1+\frac{\theta_2}{x^5}\right) \tag{5.1}$$

with the best possible constants

$$\theta_1 = 0 \quad \textit{and} \quad \theta_2 = \frac{7}{5760}. \tag{5.2}$$

Proof. We first prove the inequality (5.1) with $\theta_1 = 0$ and $\theta_2 = \frac{7}{5760}$. That is

$$\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\left(1+\frac{7}{5760x^5}\right), \qquad x>0. \tag{5.3}$$

Lambert's continued fraction [47, p. 349]

$$\tanh(z) = \cfrac{z}{1+\cfrac{z^2}{3+\cfrac{z^2}{5+\cfrac{z^2}{7+\cdots}}}}$$

is valid for all values of z. Hence for $x > 0$,

$$\frac{6x}{12x^2+1} = \frac{1/(2x)}{1+\frac{(1/(2x))^2}{3}} < \tanh\frac{1}{2x} < \frac{1/(2x)}{1+\frac{(1/(2x))^2}{3+\frac{(1/(2x))^2}{5}}} = \frac{60x^2+1}{12x(10x^2+1)}. \tag{5.4}$$

The proof of the inequality (5.3) make use of the inequality (5.4).

The lower bound in (5.3) is obtained by considering the function $f(x)$ defined for $x > 0$ by

$$f(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln x - x\ln\left(\cosh\frac{1}{2x}\right).$$

Differentiation yields

$$f'(x) = \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x} - \ln\left(\cosh\frac{1}{2x}\right) + \frac{1}{2x}\tanh\frac{1}{2x} \tag{5.5}$$

and

$$\begin{aligned} f''(x) &= \psi'(x+1) - \psi'\left(x+\frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3}\left(\tanh\frac{1}{2x}\right)^2 \\ &> \psi'(x+1) - \psi'\left(x+\frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3}\left(\frac{6x}{12x^2+1}\right)^2 =: g(x), \end{aligned}$$

by applying the left-hand inequality of (5.4). By using the recurrence formula

$$\psi'(x+1) = \psi'(x) - \frac{1}{x^2}, \tag{5.6}$$

we find that

$$g(x) - g(x+1) = \frac{h(x)}{4x^3(12x^2+1)^2(x+1)^3(2x+1)^2(12x^2+24x+13)^2}$$

with

$$\begin{aligned} h(x) &= 1540909 + 8983929(x-1) + 22585735(x-1)^2 + 32006952(x-1)^3 \\ &\quad + 27981444(x-1)^4 + 15459984(x-1)^5 + 5274000(x-1)^6 \\ &\quad + 1016064(x-1)^7 + 84672(x-1)^8. \end{aligned}$$

Hence, for $x \geq 1$,

$$g(x) > g(x+1) \quad \text{and} \quad g(x) > g(x+n).$$

Therefore, for $x \geq 1$,

$$g(x) > \lim_{n\to\infty} g(x+n) = 0 \quad \text{and} \quad f''(x) > 0.$$

We then obtain that

$$f'(x) < \lim_{t\to\infty} f'(t) = 0 \quad \text{for} \quad x \geq 1. \tag{5.7}$$

We now show that (5.7) is also valid for $0 < x \leq 1$. It follows from (5.5) that

$$-f'(x) = y_1(x) + y_2(x)$$

with

$$y_1(x) = \ln\left(\cosh\frac{1}{2x}\right) - \frac{1}{2x}\tanh\frac{1}{2x}$$

and

$$y_2(x) = -\psi(x+1) + \psi\left(x+\frac{1}{2}\right) + \frac{1}{2x}.$$

Differentiation yields

$$y_1'(x) = \frac{1}{4x^3\left(\cosh\frac{1}{2x}\right)^2} > 0.$$

Using the following representations:

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right)\mathrm{d}t$$

in [1, p. 259, 6.3.21] and

$$\frac{1}{x}=\int_0^\infty e^{-xt}\mathrm{d}t,$$

we conclude that

$$y_2(x)=\int_0^\infty\left(\frac{1}{2}-\frac{1}{e^{t/2}+1}\right)e^{-xt}\mathrm{d}t$$

and

$$y_2'(x)=-\int_0^\infty\left(\frac{1}{2}-\frac{1}{e^{t/2}+1}\right)te^{-xt}\mathrm{d}t<0.$$

Let $0\le r\le x\le s\le 1$. Since $y_1(x)$ is increasing and $y_2(x)$ is decreasing, we obtain

$$-f'(x)\ge y_1(r)+y_2(s)=:\sigma_1(r,s).$$

We divide the interval $[0,1]$ into 100 subintervals:

$$[0,1]=\bigcup_{k=0}^{99}\left[\frac{k}{100},\frac{k+1}{100}\right]>0\quad\text{for}\quad k=0,1,2,\ldots,99.$$

By direct computation we get

$$\sigma_1\left(\frac{k}{100},\frac{k+1}{100}\right)>0\quad\text{for}\quad k=0,1,2,\ldots,99.$$

Hence,

$$-f'(x)>0\quad\text{for}\quad x\in\left[\frac{k}{100},\frac{k+1}{100}\right]\quad\text{and}\quad k=0,1,2,\ldots,99.$$

This implies that $f'(x)$ is negative on $(0,1]$.

We then obtain that for all $x>0$,

$$f(x)>\lim_{t\to\infty}f(t)=0.$$

This means that the first inequality in (5.3) holds for $x>0$.

The upper bound in (5.3) is obtained by considering the function $u(x)$ defined for $x>0$ by

$$u(x)=\ln\Gamma(x+1)-\ln\Gamma\left(x+\frac{1}{2}\right)-\frac{1}{2}\ln x-x\ln\left(\cosh\frac{1}{2x}\right)-\ln\left(1+\frac{7}{5760x^5}\right).$$

Differentiation yields

$$u'(x)=\psi(x+1)-\psi\left(x+\frac{1}{2}\right)-\frac{1}{2x}-\ln\left(\cosh\frac{1}{2x}\right)+\frac{1}{2x}\tanh\frac{1}{2x}+\frac{35}{x(5760x^5+7)} \tag{5.8}$$

and

$$\begin{aligned}u''(x)&=\psi'(x+1)-\psi'\left(x+\frac{1}{2}\right)+\frac{1}{2x^2}-\frac{1}{4x^3}+\frac{1}{4x^3}\left(\tanh\frac{1}{2x}\right)^2-\frac{35(34560x^5+7)}{x^2(5760x^5+7)^2}\\&<\psi'(x+1)-\psi'\left(x+\frac{1}{2}\right)+\frac{1}{2x^2}-\frac{1}{4x^3}\\&\quad+\frac{1}{4x^3}\left(\frac{60x^2+1}{12x(10x^2+1)}\right)^2-\frac{35(34560x^5+7)}{x^2(5760x^5+7)^2}=:v(x),\end{aligned}$$

by applying the right-hand inequality of (5.4). By using the recurrence formula (5.6), we find that

$$v(x) - v(x+1) = -\frac{w_1(x)}{576 w_2(x)}$$

with

$$w_1(x) = 868270269104794924464036897 + 1876549999296938219210757164 1(x-1) + \cdots + 14820540300656640000000(x-1)^{30}$$

has all coefficients positive, and

$$w_2(x) = x^5(10x^2+1)^2(5760x^5+7)^2(x+1)^5(2x+1)^2(10x^2+20x+11)^2 \times (5760x^5+28800x^4+57600x^3+57600x^2+28800x+5767)^2.$$

Hence, for $x \geq 1$,

$$v(x) < v(x+1) \quad \text{and} \quad v(x) < v(x+n).$$

Therefore, for $x \geq 1$,

$$v(x) < \lim_{n\to\infty} v(x+n) = 0 \quad \text{and} \quad u''(x) < 0.$$

We then obtain that

$$u'(x) > \lim_{t\to\infty} u'(t) = 0 \quad \text{for} \quad x \geq 1. \tag{5.9}$$

We now show that (5.9) is also valid for $0 < x \leq 1$. It follows from (5.8) that

$$u'(x) = y_3(x) + y_4(x),$$

where $y_3(x) = -y_2(x)$ and

$$y_4(x) = -\ln\left(\cosh\frac{1}{2x}\right) + \frac{1}{2x}\tanh\frac{1}{2x} + \frac{35}{x(5760x^5+7)}.$$

Differentiation yields

$$y_4'(x) = -\frac{1}{4x^3\left(\cosh\frac{1}{2x}\right)^2} - \frac{35(34560x^5+7)}{x^2(5760x^5+7)^2} < 0.$$

Let $0 \leq r \leq x \leq s \leq 1$. Since $y_3(x)$ is increasing and $y_4(x)$ is decreasing, we obtain

$$u'(x) \geq y_3(r) + y_4(s) =: \sigma_2(r,s).$$

The same as above, we divide the interval $[0,1]$ into 100 subintervals. By direct computation we get

$$\sigma_2\left(\frac{k}{100}, \frac{k+1}{100}\right) > 0 \quad \text{for} \quad k = 0,1,2,\ldots,99.$$

Hence,

$$u'(x) > 0 \quad \text{for} \quad x \in \left[\frac{k}{100}, \frac{k+1}{100}\right] \quad \text{and} \quad k = 0,1,2,\ldots,99.$$

This implies that $u'(x)$ is positive on $(0,1]$.

We then obtain that for all $x > 0$,

$$u(x) < \lim_{t\to\infty} u(t) = 0.$$

This means that the second inequality in (5.3) holds for $x > 0$.

The inequality (5.1) can be written as

$$\theta_1 < x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] < \theta_2, \qquad x > 0.$$

We find that

$$\theta_1 \le \lim_{x\to 0^+} x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] = 0$$

and

$$\lim_{x\to\infty} x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] = \frac{7}{5760} \le \theta_2.$$

Hence, inequality (5.1) holds with the best possible constants given in equation (5.2). The proof of Theorem 5 is complete. □

Remark 4. *From* (5.3)*, we derive new inequality for the constant* π*:*

$$\left(\frac{1}{\sqrt{n}\left(\cosh\frac{1}{2n}\right)^n\left(1+\frac{7}{5760n^5}\right)} \frac{(2n)!!}{(2n-1)!!} \right)^2 < \pi < \left(\frac{1}{\sqrt{n}\left(\cosh\frac{1}{2n}\right)^n} \frac{(2n)!!}{(2n-1)!!} \right)^2 \tag{5.10}$$

for $n \in \mathbb{N}$.

Theorem 6. *For* $x \ge 1$,

$$\sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1-\frac{31}{51840x^5}\right) < \Gamma\left(x+\frac{1}{2}\right) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}. \tag{5.11}$$

Proof. From the well-known continued fraction for ψ' (see [47, p. 373])

$$\psi'\left(x+\frac{1}{2}\right) = \frac{1}{x+\frac{a_1}{x+\frac{a_2}{x+\cdots}}}, \qquad x > 0,$$

where

$$a_p = \frac{p^4}{4(2p-1)(2p+1)}, \qquad p = 1, 2, \ldots,$$

we find that $x > 0$,

$$\frac{20x(84x^2+71)}{3(560x^4+520x^2+27)} = \frac{1}{x+\frac{\frac{1}{12}}{x+\frac{\frac{4}{15}}{x+\frac{\frac{81}{140}}{x}}}} < \psi'\left(x+\frac{1}{2}\right) < \frac{1}{x+\frac{\frac{1}{12}}{x+\frac{\frac{4}{15}}{x}}} = \frac{4(15x^2+4)}{3x(20x^2+7)}. \tag{5.12}$$

The proof of the inequality (5.11) make use of the inequalities (2.22) and (5.12).

The upper bound in (5.11) is obtained by considering the function $U(x)$ defined for $x \ge 1$ by

$$U(x) = \ln\Gamma\left(x+\frac{1}{2}\right) - \ln(\sqrt{2\pi}) - x\ln x + x - \frac{x}{2}\ln(2x) - \frac{x}{2}\ln\left(\tanh\frac{1}{2x}\right).$$

Differentiation yields

$$U'(x) = \psi\left(x+\frac{1}{2}\right) - \frac{3}{2}\ln x - \frac{1}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\left(\tanh\frac{1}{2x}\right) + \frac{1}{2x\sinh\frac{1}{x}}.$$

Differentiating $U'(x)$ and applying the inequalities (2.22) and (5.12) yield

$$\begin{aligned}
U''(x) &= \psi'\left(x+\frac{1}{2}\right) + \frac{\cosh\frac{1}{x}}{x^3(\cosh\frac{2}{x}-1)} - \frac{3}{2x} \\
&< \frac{4(15x^2+4)}{3x(20x^2+7)} + \frac{1}{2x} + \frac{1}{12x^3} - \frac{7}{240x^5} + \frac{31}{6048x^7} - \frac{3}{2x} \\
&= -\frac{3074x^2-1085}{30240x^7(20x^2+7)} < 0 \quad \text{for} \quad x \geq 1.
\end{aligned}$$

We then obtain that for $x \geq 1$,

$$U'(x) > \lim_{t\to\infty} U'(t) = 0 \Longrightarrow U(x) < \lim_{t\to\infty} U(t) = 0.$$

This means that the second inequality in (5.11) holds for $x \geq 1$.

The lower bound in (5.11) is obtained by considering the function $F(x)$ defined for $x \geq 1$ by

$$\begin{aligned}
F(x) &= \ln\Gamma\left(x+\frac{1}{2}\right) - \ln(\sqrt{2\pi}) - x\ln x + x - \frac{x}{2}\ln(2x) \\
&\quad - \frac{x}{2}\ln\left(\tanh\frac{1}{2x}\right) - \ln\left(1-\frac{31}{51840x^5}\right).
\end{aligned}$$

Differentiation yields

$$\begin{aligned}
F'(x) &= \psi\left(x+\frac{1}{2}\right) - \frac{3}{2}\ln x - \frac{1}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\left(\tanh\frac{1}{2x}\right) \\
&\quad + \frac{1}{2x\sinh\frac{1}{x}} - \frac{155}{x(51840x^5-31)}.
\end{aligned}$$

Differentiating $F'(x)$ and applying the inequalities (2.22) and (5.12) yield

$$\begin{aligned}
F''(x) &= \psi'\left(x+\frac{1}{2}\right) + \frac{\cosh\frac{1}{x}}{x^3(\cosh\frac{2}{x}-1)} \\
&\quad - \frac{8062156800x^{11} - 96422400x^5 - 9642240x^6 + 2883x + 9610}{2x^2(51840x^5-31)^2} \\
&> \frac{20x(84x^2+71)}{3(560x^4+520x^2+27)} + \frac{1}{2x} + \frac{1}{12x^3} - \frac{7}{240x^5} + \frac{31}{6048x^7} - \frac{127}{172800x^9} \\
&\quad - \frac{8062156800x^{11} - 96422400x^5 - 9642240x^6 + 2883x + 9610}{2x^2(51840x^5-31)^2} \\
&= \frac{G(x)}{1209600x^9(560x^4+520x^2+27)(51840x^5-31)^2},
\end{aligned}$$

where

$$\begin{aligned} G(x) =& 35836321468874877 + 500332411677183040(x-1) \\ &+ 3243650443203650720(x-1)^2 + 12942574029209436800(x-1)^3 \\ &+ 35510225676993688800(x-1)^4 + 70870406789351495040(x-1)^5 \\ &+ 106102603246810676800(x-1)^6 + 121054817642726534400(x-1)^7 \\ &+ 105767748316645632000(x-1)^8 + 70425167657151744000(x-1)^9 \\ &+ 35177843816429875200(x-1)^{10} + 12782474258840064000(x-1)^{11} \\ &+ 3194045448118272000(x-1)^{12} + 491271286947840000(x-1)^{13} \\ &+ 35090806210560000(x-1)^{14}. \end{aligned}$$

Hence, $F''(x) > 0$ for $x \geq 1$. We then obtain that for $x \geq 1$,

$$F'(x) < \lim_{t\to\infty} F'(t) = 0 \Longrightarrow F(x) > \lim_{t\to\infty} F(t) = 0.$$

This means that the first inequality in (5.11) holds for $x \geq 1$. The proof of Theorem 6 is complete. □

Remark 5. *Some computer experiments indicate that for $x > 0$,*

$$\begin{aligned} &\sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1-\frac{\vartheta_1}{x^5}\right) \\ &\qquad < \Gamma\left(x+\frac{1}{2}\right) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1-\frac{\vartheta_2}{x^5}\right) \end{aligned} \tag{5.13}$$

with the best possible constants $\vartheta_1 = \frac{31}{51840}$ and $\vartheta_2 = 0$.

Although the double inequality (5.11) is given only for $x \geq 1$, its main utility is in the evaluation of $\Gamma\left(x+\frac{1}{2}\right)$ for large values of the argument.

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