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Extended Riemann-Liouville fractional derivative operator and its applications

Praveen Agarwal[a], Junesang Choi[b,*], R. B. Paris[c]

[a]*Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India.*
[b]*Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea.*
[c]*School of Computing, Engineering and Applied Mathematics, University of Abertay Dundee, Dundee DD1 1HG, UK.*

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Abstract

Many authors have introduced and investigated certain extended fractional derivative operators. The main object of this paper is to give an extension of the Riemann-Liouville fractional derivative operator with the extended Beta function given by Srivastava *et al.* [22] and investigate its various (potentially) useful and (presumably) new properties and formulas, for example, integral representations, Mellin transforms, generating functions, and the extended fractional derivative formulas for some familiar functions.

Keywords: Gamma function, Beta function, Riemann-Liouville fractional derivative, hypergeometric functions, fox H-function, generating functions, Mellin transform, integral representations.
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1. Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past four decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering (see, *e.g.*, [1, 9, 11, 13, 14, 25]). The review-cum-survey paper [13] is gladly recommended for the readers who would like to know some of the major documents and events in the area of fractional calculus that took place since 1974 up to 2010. In recent years, due to the above-mentioned motivation, certain

*Corresponding author
Email addresses: `goyal.praveen2011@gmail.com` (Praveen Agarwal), `junesang@mail.dongguk.ac.kr` (Junesang Choi), `r.paris@abertay.ac.uk` (R. B. Paris)

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extended fractional derivative operators associated with special functions have been actively investigated. Many authors have introduced certain extended fractional derivative operators (see, *e.g.*, [12, 20]). Recently, Srivastava *et al.* [22] introduced the following extended Beta function:

Definition 1.1. The extended beta function $\mathcal{B}_p^{(\alpha,\beta;\kappa,\mu)}(x,y)$ with $\Re(p)>0$ is defined by

$$\mathcal{B}_p^{(\alpha,\beta;\kappa,\mu)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t^{\kappa}(1-t)^{\mu}}\right)dt, \tag{1.1}$$

where

$$\kappa\geq 0,\ \mu\geq 0,\ \min\{\Re(\alpha),\Re(\beta)\}>0,\ \Re(x)>-\Re(\kappa\alpha),\ \Re(y)>-\Re(\mu\alpha).$$

Remark 1.2. Various properties of the function (1.1) have been studied by Luo *et al.* [12]. The special case of (1.1) when $p=0$ is seen to immediately reduce to the familiar beta function $B(x,y)$ ($\min\{\Re(x),\Re(y)\}>0$) (see, e.g., [23, Section 1.1]). Other various special cases of (1.1) obtained by specializing the parameters have been studied by many authors (see [5, 6, 7, 16, 21]).

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}^-$, and $\mathbb{N}$ be sets of complex numbers, positive real numbers, negative integers, and positive integers, respectively, and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and $\mathbb{Z}_0^-:=\mathbb{Z}^-\cup\{0\}$. We also recall to use the following definition [22].

Definition 1.3. The extended Gauss hypergeometric function is defined by

$$\begin{gathered}F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z):=\sum_{n=0}^{\infty}(a)_n\,\frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\,\frac{z^n}{n!}\\ \left(|z|<1;\ \min\{\Re(\alpha),\Re(\beta),\Re(\kappa),\Re(\mu)\}>0;\ \Re(c)>\Re(b)>0;\ \Re(p)\geqq 0\right),\end{gathered} \tag{1.2}$$

where $B(u,v)$ is the familiar Beta function defined by (see, e.g., [23, p. 8])

$$B(u,\,v)=\begin{cases}\displaystyle\int_0^1 t^{u-1}(1-t)^{v-1}\,dt & (\Re(u)>0;\ \Re(v)>0)\\[2ex] \displaystyle\frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)} & \left(u,\,v\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).\end{cases} \tag{1.3}$$

Here Γ denotes the Euler's Gamma function (see, e.g., [23, Section 1.1]).

The special case of (1.2) when $p=0$ is noted to reduce to the ordinary Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ (see, *e.g.*, [23, Section 1.5]).

Motivated by the various extensions of the fractional derivative operators which have recently been considered by many authors, here, we aim to introduce an extended Riemann-Liouville fractional derivative operator involving the generalized hypergeometric-type function $F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z)$ (1.2) and investigate some of its properties. Next, extensions of some extended hypergeometric functions and their integral representations are presented by using the extended Riemann-Liouville fractional derivative operator. The linear and bilinear generating relations for the extended hypergeometric functions, their representations in terms of the Fox H-function and Mellin transforms of the extended fractional derivatives are also determined. Finally, we define the extended fractional derivative operator in a different form with respect to an arbitrary, regular and univalent function based on the Cauchy integral formula.

2. Extended Hypergeometric Functions

In this section we define the extended Gauss hypergeometric function $F_{p;\kappa,\mu}$, the Appell hypergeometric functions $F_{1,p;\kappa,\mu}$, $F_{2,p;\kappa,\mu}$ and the Lauricella hypergeometric function $F^{D}_{3,p;\kappa,\mu}$ and then obtain their integral representations involving the extended Gauss hypergeometric function (1.2). Throughout this section we assume $m \in \mathbb{N}_0$.

Definition 2.1. A further extension of the extended Gauss hypergeometric function $F_p^{(\alpha,\beta;\kappa,\mu)}$ is defined by

$$F_{p;\kappa,\mu}(a,b;c;z;m) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{B_p^{\alpha,\beta;\kappa,\mu}(b+n, c-b+m)}{B(b+n, c-b+m)} \frac{z^n}{n!} \tag{2.1}$$
$$\big(p \geq 0;\ \Re(\kappa) > 0;\ \Re(\mu) > 0;\ m < \Re(b) < \Re(c);\ |z| < 1\big).$$

Definition 2.2. A further extension of the extended Appell hypergeometric function F_1 is defined by

$$\begin{aligned} &F_{1,p;\kappa,\mu}(a,b,c;d;x,y;m) \\ &:= \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(d)_{n+k}} \frac{B_p^{\alpha,\beta;\kappa,\mu}(a+n+k, d-a+m)}{B(a+n+k, d-a+m)} \frac{x^n}{n!} \frac{y^k}{k!} \end{aligned} \tag{2.2}$$
$$\big(p \geq 0;\ \Re(\kappa) > 0;\ \Re(\mu) > 0;\ m < \Re(a) < \Re(d);\ |x| < 1;\ |y| < 1\big).$$

Definition 2.3. A further extension of the Appell hypergeometric function F_2 is defined by

$$\begin{aligned} F_{2,p;\kappa,\mu}(a,b,c;d,e;x,y;m) := \sum_{n,k=0}^{\infty} \Bigg[&\frac{(a)_{n+k}(b)_n(c)_k}{(d)_n(e)_k} \\ \times &\frac{B_p^{\alpha,\beta;\kappa,\mu}(b+n, d-b+m)}{B(b+n, d-b+m)} \frac{B_p^{\alpha,\beta;\kappa,\mu}(c+k, e-c+m)}{B(c+k, e-c+m)} \frac{x^n z^k}{n!k!} \Bigg] \end{aligned} \tag{2.3}$$
$$\big(p \geq 0;\ \Re(\kappa) > 0;\ \Re(\mu) > 0;\ m < \Re(b) < \Re(d);\ m < \Re(c) < \Re(e);\ |x| + |y| < 1\big).$$

Definition 2.4. A further extension of the Lauricella hypergeometric function F_D^3 is defined by

$$\begin{aligned} &F^3_{D,p;\kappa,\mu}(a,b,c,d;e;x,y,z;m) \\ &:= \sum_{n,k,r=0}^{\infty} \frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(e)_{n+k+r}} \frac{B_p^{\alpha,\beta;\kappa,\mu}(a+n+k+r, e-a+m)}{B(a+n+k+r, e-a+m)} \frac{x^n}{n!} \frac{y^k}{k!} \frac{z^r}{r!} \end{aligned} \tag{2.4}$$
$$\big(p \geq 0;\ \Re(\kappa) > 0;\ \Re(\mu) > 0;\ m < \Re(a) < \Re(e);\ |x| < 1;\ |y| < 1;\ |z| < 1\big).$$

It is noted that the special cases of (2.1), (2.2), (2.3), and (2.4) when $p = 0$ and $m = 0$ reduce to the well-known Gauss hypergeometric function ${}_2F_1$, the Appell functions F_1, F_2, and the Lauricella function F_D^3, respectively (see, *e.g.*, [24, p. 53 and p. 61]).

We present certain integral representations of the extended hypergeometric functions (2.1), (2.2), (2.3) and (2.4) by the following theorem.

Theorem 2.5. *The following integral representations for the extended hypergeometric functions $F_{p;\kappa,\mu}$, $F_{1,p;\kappa,\mu}$, $F_{2,p;\kappa,\mu}$ and $F_{D,p;\kappa,\mu}$ hold true:*

$$\begin{aligned} &F_{p;\kappa,\mu}(a,b;c;z;m) = \frac{1}{B(b, c-b+m)} \\ &\times \int_0^1 \left\{ t^{b-1}(1-t)^{c-b+m-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^{\kappa}(1-t)^{\mu}}\right) {}_2F_1(a, c+n; c; zt) \right\} dt; \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
F_{1,p;\kappa,\mu}(a,b,c;d;x,y;m) = \frac{1}{B(a,d-a+m)} \int_0^1 \Big\{ & t^{a-1}(1-t)^{d-a+m-1} \\
& \times {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa(1-t)^\mu}\right) F_1(d+m,b,c;d;xt,yt) \Big\} dt;
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
& F_{2,p;\kappa,\mu}(a,b,c;d,e;x,y;m) \\
&= \frac{1}{B(b,d-b+m)B(c,e-c+m)} \int_0^1 \int_0^1 \Big\{ t^{b-1}(1-t)^{d-b+m-1} \\
&\qquad \times u^{c-1}(1-u)^{e-c+m-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa(1-t)^\mu}\right) \\
&\qquad \times {}_1F_1\left(\alpha;\beta;-\frac{p}{u^\kappa(1-u)^\mu}\right) F_2(a,d+m,e+m;d,e;xt,yu) \Big\} dt du;
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
& F^3_{D,p;\kappa,\mu}(a,b,c,d;e;x,y,z;m) \\
&= \frac{1}{B(a,e-a+m)} \int_0^1 \Big\{ t^{a-1}(1-t)^{e-a+m-1} \\
&\qquad \times {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa(1-t)^\mu}\right) F^3_D(e+m,b,c,d;e;xt,yt,zt) \Big\} dt.
\end{aligned}
\tag{2.8}
$$

Proof. The integral representations (2.5)–(2.8) can be obtained directly by replacing the function $B_p^{(\alpha,\beta;\kappa,\mu)}$ with its integral representation in (2.1)–(2.4), respectively. □

3. Extended Riemann-Liouville Fractional Derivative Operator

In this section, we consider the extended Riemann-Liouville type fractional derivative operator and then determine the extended fractional derivatives of some elementary functions. For this purpose, we begin by recalling the classical Riemann-Liouville fractional derivative of $f(z)$ of order ν defined by

$$
D_z^\nu f(z) := \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{-\nu-1} f(t)\, dt \quad (\Re(\nu) < 0),
$$

where the integration path is a line from 0 to z in the complex t-plane. When $\Re(\nu) \geq 0$, let $m \in \mathbb{N}$ be the smallest integer greater than $\Re(\nu)$ and so $m-1 \leq \Re(\nu) < m$. Then the Riemann-Liouville fractional derivative of $f(z)$ of order ν is defined by

$$
\begin{aligned}
D_z^\nu f(z) &:= \frac{d^m}{dz^m} D_z^{\nu-m} f(z), \\
&= \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(m-\nu)} \int_0^z (z-t)^{m-\nu-1} f(t)\, dt \right\}.
\end{aligned}
$$

The fractional integral and derivative operators involving various special functions have found significant importance and applications in various areas, for example, mathematical physics as well as mathematical analysis. In recent years, many authors have developed various extended fractional derivative formulas of Riemann-Liouville type. Here, we present some new extended Riemann-Liouville type fractional derivative formulas.

Definition 3.1. The extended Riemann-Liouville fractional derivative of $f(z)$ of order ν is defined by

$$
\begin{aligned}
D_z^{\nu,p;\kappa,\mu} f(z) := \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{-\nu-1} f(t) {}_1F_1\left(\alpha;\beta;-\frac{pz^{\kappa+\mu}}{t^\kappa(z-t)^\mu}\right) dt \\
\big(\Re(\nu) < 0;\ \Re(p) > 0;\ \Re(\kappa) > 0;\ \Re(\mu) > 0\big).
\end{aligned}
\tag{3.1}
$$

When $\Re(\nu) \geq 0$, let $m \in \mathbb{N}$ be the smallest integer greater than $\Re(\nu)$ and so $m-1 \leq \Re(\nu) < m$. Then the extended Riemann-Liouville fractional derivative of $f(z)$ of order ν is defined by

$$\begin{aligned} D_z^{\nu,p;\kappa,\mu} f(z) &:= \frac{d^m}{dz^m} D_z^{\nu-m,p;\kappa,\mu} f(z) \\ &= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\nu)}\int_0^z (z-t)^{m-\nu-1} f(t)\,{}_1F_1\left(\alpha;\beta;-\frac{pz^{\kappa+\mu}}{t^\kappa (z-t)^\mu}\right)dt\right\} \\ &\qquad \big(\Re(p)>0;\ \Re(\kappa)>0;\ \Re(\mu)>0\big). \end{aligned} \tag{3.2}$$

Remark 3.2. The special case of (3.1) and (3.2) when $p=0$ becomes the classical Riemann-Liouville fractional derivative. The special case of (3.1) and (3.2) when $\alpha=\beta$ and $\kappa=\mu=1$ is seen to reduce to the known one [20].

We consider the extended fractional derivative of the function z^λ.

Theorem 3.3. *Let $m-1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$ and $\Re(\nu) < \Re(\lambda)$. Then we have*

$$D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)}{\Gamma(\lambda-\nu+1)B(\lambda+1,m-\nu)} z^{\lambda-\nu}. \tag{3.3}$$

Proof. Applying (3.2) in Definition 3.1 to the function z^λ, we have

$$D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\} = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\nu)}\int_0^z (z-t)^{m-\nu-1}\, t^\lambda\,{}_1F_1\left(\alpha;\beta;-\frac{pz^{\kappa+\mu}}{t^\kappa (z-t)^\mu}\right)dt\right\}.$$

Setting $t=zu$ in this expression, we get

$$\begin{aligned} D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\} &= \left(\frac{d^m}{dz^m} z^{m+\lambda-\nu}\right) \\ &\times \frac{1}{\Gamma(m-\nu)}\int_0^1 (1-u)^{m-\nu-1}\, u^{\lambda+1-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{u^\kappa (1-u)^\mu}\right)du. \end{aligned}$$

Considering

$$\frac{d^m}{dz^m} z^{m+\lambda-\nu} = \frac{\Gamma(1+\lambda-\nu+m)}{\Gamma(1+\lambda-\nu)}\, z^{\lambda-\nu},$$

in view of (1.1) and the second identity of (1.3), we are led to the desired result. □

We apply the extended Riemann-Liouville fractional derivative to a function $f(z)$ analytic at the origin.

Theorem 3.4. *Let $m-1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^\infty a_n z^n$ $(|z|<\rho)$ for some $\rho \in \mathbb{R}^+$. Then we have*

$$D_z^{\nu,p;\kappa,\mu}\{f(z)\} = \sum_{n=0}^\infty a_n D_z^{\nu,p;\kappa,\mu}\{z^n\}.$$

Proof. Applying (3.2) in Definition 3.1 to the function $f(z)$ with its series expansion, we have

$$\begin{aligned} &D_z^{\nu,p;\kappa,\mu}\{f(z)\} \\ &= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\nu)}\int_0^z (z-t)^{m-v-1}\,{}_1F_1\left(\alpha;\beta;-\frac{pz^{\kappa+\mu}}{t^\kappa (z-t)^\mu}\right)\sum_{n=0}^\infty a_n\, t^n\, dt\right\}. \end{aligned}$$

Since the power series converges uniformly on any closed disk centered at the origin with its radius smaller than ρ, so does the series on the line segment from 0 to a fixed z for $|z| < \rho$. This fact guarantees term-by-term integration as follows:

$$\begin{aligned} D_z^{\nu,p;\kappa,\mu}\{f(z)\} &= \sum_{n=0}^{\infty} a_n \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\nu)}\int_0^z (z-t)^{m-\nu-1}{}_1F_1\left(\alpha;\beta;-\frac{pz^{\kappa+\mu}}{t^\kappa (z-t)^\mu}\right) t^n\,dt\right\} \\ &= \sum_{n=0}^{\infty} a_n D_z^{\nu,p;\kappa,\mu}\{z^n\}. \end{aligned}$$

□

The following theorem is seen to immediately follow from Theorems 3.3 and 3.4.

Theorem 3.5. *Let $m-1 \le \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < \rho)$ for some $\rho \in \mathbb{R}^+$. Then we have*

$$\begin{aligned} D_z^{\nu,p;\kappa,\mu}\left\{z^{\lambda-1} f(z)\right\} &= \sum_{n=0}^{\infty} a_n D_z^{\nu,p;\kappa,\mu}\left\{z^{\lambda+n-1}\right\} \\ &= \frac{\Gamma(\lambda) z^{\lambda-\nu-1}}{\Gamma(\lambda-\nu)} \sum_{n=0}^{\infty} a_n \frac{(\lambda)_n}{(\lambda-\nu)_n} \frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+n, m-\nu)}{B(\lambda+n, m-\nu)} z^n. \end{aligned}$$

We present two subsequent theorems which may be useful to find certain generating function relations.

Theorem 3.6. *Let $m-1 \le \Re(\lambda-\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have*

$$\begin{aligned} & D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha}\right\} \\ &= \frac{\Gamma(\lambda) z^{\nu-1}}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\nu)_n} \frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+n, \nu-\lambda+m)}{B(\lambda+n, \nu-\lambda+m)} \frac{z^n}{n!} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} F_{p;\kappa,\mu}(\alpha, \lambda; \nu; z; m) \quad (|z| < 1;\ \alpha \in \mathbb{C}). \end{aligned} \tag{3.4}$$

Proof. Using the generalized binomial theorem:

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n \quad (|z| < 1;\ \alpha \in \mathbb{C})$$

and applying Theorems 3.3 and 3.4, we obtain

$$\begin{aligned} D_z^{\lambda-\nu,p;\kappa,\mu}\{z^{\lambda-1}(1-z)^{-\alpha}\} &= D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n \frac{z^n}{n!}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda+n-1}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{\Gamma(\lambda+n)}{\Gamma(\nu+n)} \frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+n, m-\lambda+\nu)}{B(\lambda+n, m-\lambda+\nu)} z^{\nu+n-1} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\nu)_n} \frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+n, m-\lambda+\nu)}{B(\lambda+n, m-\lambda+\nu)} \frac{z^n}{n!} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} F_{p;\kappa,\mu}(\alpha, \lambda; \nu; z; m). \end{aligned}$$

□

Theorem 3.7. *Let $m-1 \leq \Re(\lambda-\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have*

$$\begin{aligned} &D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} \sum_{n,k=0}^{\infty} \frac{(\lambda)_{n+k}(\alpha)_n(\beta)_k}{(\nu)_{n+k}} \frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+n+k,\nu-\lambda+m)}{B(\lambda+n+k,\nu-\lambda+m)} \frac{(az)^n}{n!} \frac{(bz)^k}{k!} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} F_{1,p;\kappa,\mu}(\lambda,\alpha,\beta;\nu;az;bz;m) \\ &\quad (|az|<1;\ |bz|<1;\ a,\ b,\ \alpha,\ \beta \in \mathbb{C}). \end{aligned} \tag{3.5}$$

Proof. Using the binomial theorems for $(1-az)^{-\alpha}$ and $(1-bz)^{-\beta}$, as in the proof of (3.6), we can prove (3.5). The details of its proof are omitted. □

Similarly as in Theorems 3.6 and 3.7, we can obtain the following expression.

Theorem 3.8. *Let $m-1 \leq \Re(\lambda-\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have*

$$\begin{aligned} &D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} \sum_{n,k,r=0}^{\infty} \frac{(\lambda)_{n+k+r}(\alpha)_n(\beta)_k(\gamma)_r}{(\nu)_{n+k+r}} \\ &\quad \times \frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+n+k+r,\nu-\lambda+m)}{B(\lambda+n+k+r,\nu-\lambda+m)} \frac{(az)^n}{n!} \frac{(bz)^k}{k!} \frac{(cz)^r}{r!} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1} F^3_{D,p;\kappa,\mu}(\lambda,\alpha,\beta,\gamma;\nu;az;bz;cz;m) \\ &\quad (|az|<1;\ |bz|<1;\ |cz|<1;\ a,\ b,\ \alpha,\ \beta,\ \gamma \in \mathbb{C}). \end{aligned} \tag{3.6}$$

Theorem 3.9. *Let*

$$m-1 \leq \Re(\lambda-\nu) < m < \Re(\lambda)$$

and

$$m < \Re(\beta) < \Re(\gamma)$$

for some $m \in \mathbb{N}$. Then we have

$$\begin{aligned} &D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha} F_{p;\kappa,\mu}(\alpha,\beta;\gamma;\frac{x}{1-z};m)\right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\nu-1} \sum_{n,k=0}^{\infty} \left\{ \frac{(\alpha)_{n+k}(\beta)_n(\lambda)_k}{(\gamma)_n(\nu)_k} \frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+n,\gamma-\beta+m)}{B(\beta+n,\gamma-\beta+m)} \right. \\ &\qquad \left. \times \frac{B_{p;\kappa,\mu}(\lambda+k,\nu-\lambda+m)}{B(\lambda+k,\nu-\lambda+m)} \frac{x^n z^k}{n!k!} \right\} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\nu-1} F_{2,p;\kappa,\mu}(\alpha,\beta,\lambda;\gamma,\nu;x,z;m) \\ &\quad (|x|+|z|<1;\ \alpha \in \mathbb{C}). \end{aligned} \tag{3.7}$$

Proof. Using the binomial theorem for $(1-z)^{-\alpha}$ and applying the Definition 2.1 for $F_{p;\kappa,\mu}$, we get

$$\begin{aligned}
&D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p;\kappa,\mu}(\alpha,\beta;\gamma;\frac{x}{1-z};m)\right\}\\
&=D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+n,\gamma-\beta+m)}{B(\beta+n,\gamma-\beta+m)}\left(\frac{x}{1-z}\right)^n\right\}\\
&=D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha-n}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+n,\gamma-\beta+m)}{B(\beta+n,\gamma-\beta+m)}\frac{x^n}{n!}\right\}\\
&=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+n,\gamma-\beta+m)}{B(\beta+n,\gamma-\beta+m)}\frac{x^n}{n!}D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha-n}\right\}.
\end{aligned}$$

We therefore have

$$\begin{aligned}
&D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p;\kappa,\mu}(\alpha,\beta;\gamma;\frac{x}{1-z};m)\right\}\\
&=\frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{(\alpha)_{n+k}(\beta)_n(\lambda)_k}{(\gamma)_n(\nu)_k}\right.\\
&\quad\left.\times\frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+n,\gamma-\beta+m)}{B(\beta+n,\gamma-\beta+m)}\frac{B_p^{\alpha,\beta;\kappa,\nu}(\lambda+k,\nu-\lambda+m)}{B(\lambda+k,\nu-\lambda+m)}\frac{x^nz^k}{n!k!}\right\}\\
&=\frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}F_{2,p;\kappa,\mu}(\alpha,\beta,\lambda;\gamma,\nu;x,z;m).
\end{aligned}$$

□

4. Generating Functions Involving the Extended Gauss Hypergeometric Function

In this section, we establish some linear and bilinear generating relations for the extended hypergeometric function $F_{p;\kappa,\mu}$ by using Theorems 3.6, 3.7 and 3.9.

Theorem 4.1. *Let $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$ for some $m\in\mathbb{N}$. Then we have*

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p;\kappa,\mu}(\alpha+n,\lambda;\nu;z;m)t^n=(1-t)^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\lambda;\nu;\frac{z}{1-t};m\right) \tag{4.1}$$
$$(|z|<\min\{1,|1-t|\};\ \alpha\in\mathbb{C}).$$

Proof. We start by recalling the elementary identity (see [24, p. 291] and [20, p. 1832]):

$$[(1-z)-t]^{-\alpha}=(1-t)^{-\alpha}\left(1-\frac{z}{1-t}\right)^{-\alpha}$$

and expand its left-hand side to obtain

$$(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\alpha}\left(1-\frac{z}{1-t}\right)^{-\alpha}\quad(|t|<|1-z|).$$

Multiplying both sides of the above equality by $z^{\lambda-1}$ and applying the extended Riemann-Liouville fractional derivative operator $D_z^{\lambda-\nu,p;\kappa,\mu}$ on both sides, we find

$$D_z^{\lambda-\nu,p;\kappa,\mu}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_nt^n}{n!}z^{\lambda-1}(1-z)^{-\alpha-n}\right\}=D_z^{\lambda-\nu,p;\kappa,\mu}\left\{(1-t)^{-\alpha}z^{\lambda-1}\left(1-\frac{z}{1-t}\right)^{-\alpha}\right\}.$$

Uniform convergence of the involved series makes it possible to exchange the summation and the fractional operator to give

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\alpha-n}\right\}t^n=(1-t)^{-\alpha}D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}\left(1-\frac{z}{1-t}\right)^{-\alpha}\right\}.$$

The result then follows by applying Theorem 3.6 to both sides of the last identity. □

Theorem 4.2. *Let $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$ for some $m\in\mathbb{N}$. Then we have*

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p;\kappa,\mu}(\beta-n,\lambda;\nu;z;m)t^n=(1-t)^{-\alpha}F_{1,p;\kappa,\mu}\left(\beta,\alpha,\lambda;\nu;z;\frac{-zt}{1-t};m\right)$$

$$(\alpha,\ \beta\in\mathbb{C};\ |z|<1;\ |t|<|1-z|;\ |z||t|<|1-t|).$$

Proof. Considering the following identity (see [24, p. 291] and [7, p. 595]):

$$[1-(1-z)t]^{-\alpha}=(1-t)^{-\alpha}\left(1+\frac{zt}{1-t}\right)^{-\alpha}$$

and expanding its left-hand side as a power series, we get

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}(1-z)^nt^n=(1-t)^{-\alpha}\left(1-\frac{-zt}{1-t}\right)^{-\alpha}\qquad(|t|<|1-z|).$$

Multiplying both sides by $z^{\lambda-1}(1-z)^{-\beta}$ and applying the definition of the extended Riemann-Liouville fractional derivative operator $D_z^{\lambda-\nu,p;\kappa,\mu}$ on both sides, we find

$$\begin{aligned}&D_z^{\lambda-\nu,p;\kappa,\mu}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^{\lambda-1}(1-z)^{-\beta}(1-z)^nt^n\right\}\\&=D_z^{\lambda-\nu,p;\kappa,\mu}\left\{(1-t)^{-\alpha}z^{\lambda-1}(1-z)^{-\beta}\left(1-\frac{-zt}{1-t}\right)^{-\alpha}\right\}.\end{aligned}$$

The given conditions are found to allow us to exchange the order of the summation and the fractional derivative to yield

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\beta+n}\right\}t^n\\&=(1-t)^{-\alpha}D_z^{\lambda-\nu,p;\kappa,\mu}\left\{z^{\lambda-1}(1-z)^{-\beta}\left(1-\frac{-zt}{1-t}\right)^{-\alpha}\right\}.\end{aligned}$$

Finally the result follows by using Theorems 3.6 and 3.7. □

Theorem 4.3. *Let*

$$m-1<\Re(\beta-\gamma)<m<\Re(\beta)$$

and

$$m<\Re(\lambda)<\Re(\nu)$$

for some $m\in\mathbb{N}$. Then we have

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p;\kappa,\mu}(\alpha+n,\lambda;\nu;z;m)F_{p;\kappa,\mu}(-n,\beta;\gamma;u;m)=F_{2,p;\kappa,\mu}\left(\alpha,\lambda,\beta;\nu,\gamma;z,\frac{-ut}{1-t};m\right)$$

$$\left(\alpha\in\mathbb{C};\ |z|<1;\ \left|\frac{1-u}{1-z}t\right|<1;\ \left|\frac{z}{1-t}\right|+\left|\frac{ut}{1-t}\right|<1\right).$$

Proof. Replacing t by $(1-u)t$ in (4.1) and multiplying both sides of the resulting identity by $u^{\beta-1}$ gives

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p;\kappa,\mu}(\alpha+n,\lambda;\nu;z;m)u^{\beta-1}(1-u)^n t^n \\ = u^{\beta-1}[1-(1-u)t]^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\lambda;\nu;\frac{z}{1-(1-u)t};m\right).$$

Applying the fractional derivative $D_u^{\lambda-\nu,p;\kappa,\mu}$ to both sides of the resulting identity and changing the order of the summation and the fractional derivative yields

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p;\kappa,\mu}(\alpha+n,\lambda;\nu;z;m)D_u^{\beta-\gamma,p;\kappa,\mu}\left\{u^{\beta-1}(1-u)^n\right\}t^n \\ = D_u^{\beta-\gamma,p;\kappa,\mu}\left\{u^{\beta-1}[1-(1-u)t]^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\lambda;\nu;\frac{z}{1-(1-u)t};m\right)\right\} \\ (|(1-u)t|<1;\ |ut|<|1-t|).$$

The last identity can be written as follows:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p;\kappa,\mu}(\alpha+n,\lambda;\nu;z;m)D_u^{\beta-\gamma,p;\kappa,\mu}\left\{u^{\beta-1}(1-u)^n\right\}t^n \\ = D_u^{\beta-\gamma,p;\kappa,\mu}\left\{u^{\beta-1}\left[1-\frac{-ut}{1-t}\right]^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\lambda;\nu;\frac{z}{1-\frac{-ut}{1-t}};m\right)\right\}.$$

Finally the use of Theorems 3.6 and 3.9 in the resulting identity is seen to give the desired result. □

5. Mellin Transforms and Further Results

In this section, we first obtain the Mellin transform of the extended Beta function given by (1.1) and use this transform to find the Mellin transform of the extended Riemann-Liouville fractional derivative operator. We then apply the extended fractional derivative operator (6.2) to the familiar functions e^z, ${}_2F_1$ and represent z^λ in terms of the Fox H-function.

The following three theorems pertain to the Mellin transforms of the extended Beta function and Riemann-Liouville fractional derivatives of two functions.

Theorem 5.1. *Let $\Re(s)>0$, $\Re(x+\kappa\, s)>0$, $\Re(y+\mu\, s)>0$ and $p>0$. Then the following Mellin transform holds true:*

$$\mathfrak{M}\left[B_p^{\alpha,\beta;\kappa,\mu}(x,y):s\right]=B(x+\kappa s,y+\mu s)\,\Gamma^{(\alpha,\beta)}(s),$$

where (see [20])

$$\Gamma^{(\alpha,\beta)}(s):=\int_0^\infty b^{s-1}{}_1F_1(\alpha;\beta;-b)\,db \\ (\Re(s)>0,\ \Re(\alpha+s)>0,\ \Re(\beta+s)>0). \tag{5.1}$$

Proof. Taking the Mellin transform of $B_p^{\alpha,\beta;\kappa,\mu}(x,y)$, we find

$$\mathfrak{M}\left[B_p^{\alpha,\beta;\kappa,\mu}(x,y):s\right] \\ =\int_0^\infty p^{s-1}\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa(1-t)^\mu}\right)dt\,dp. \tag{5.2}$$

Since, under the given conditions,

$$F(t) := \int_0^\infty p^{s-1}\, t^{x-1}\, (1-t)^{y-1}\, {}_1F_1\left(\alpha; \beta; -\frac{p}{t^\kappa\,(1-t)^\mu}\right) dp$$

converges for each point $t \in (0,\, 1)$ converges uniformly on $(0,\, 1)$, the order of integrations in (5.2) can be interchanged. We therefore have

$$\begin{aligned} &\mathfrak{M}\left[B_p^{\alpha,\beta;\kappa,\mu}(x,y) : s\right] \\ &= \int_0^1 t^{x-1}\,(1-t)^{y-1}\left\{\int_0^\infty p^{s-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa\,(1-t)^\mu}\right) dp\right\} dt. \end{aligned} \tag{5.3}$$

Setting $\omega = \frac{p}{t^\kappa (1-t)^\mu}$, we have

$$\begin{aligned} &\mathfrak{M}\left[B_p^{\alpha,\beta;\kappa,\mu}(x,y) : s\right] \\ &= \int_0^1 t^{x+\kappa s-1}\,(1-t)^{y+\mu s-1}\left\{\int_0^\infty \omega^{s-1}\,{}_1F_1\left(\alpha;\beta;-\omega\right) d\omega\right\} dt. \end{aligned} \tag{5.4}$$

Hence it is easy to see the desired result. □

Theorem 5.2. *Let $\Re(s) > 0$, $\Re(x + \kappa\, s) > 0$, $\Re(y + \mu\, s) > 0$, $p > 0$, and $\Re(\lambda) > m - 1$ for some $m \in \mathbb{N}$. Then we have*

$$\mathfrak{M}\left[D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\} : s\right] = \frac{\Gamma(\lambda+1)\Gamma^{(\alpha,\beta)}(s)\, B(m-\nu+s, \lambda-m+s+1)}{\Gamma(\lambda-\nu+1)\, B(m-\nu,\lambda+1)} z^{\lambda-\nu}.$$

Proof. Taking the Mellin transform and using Theorem 3.3, we have

$$\begin{aligned} \mathfrak{M}\left[D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\} : s\right] &= \int_0^\infty p^{s-1} D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\} dp \\ &= \int_0^\infty p^{s-1}\frac{\Gamma(\lambda+1)B_p^{\alpha,\beta;\kappa,\mu}(m-\nu,\lambda+1)}{\Gamma(\lambda-\nu+1)B(m-\nu,\lambda+1)} z^{\lambda-\nu} dp \\ &= \frac{\Gamma(\lambda+1)z^{\lambda-\nu}}{\Gamma(\lambda-\nu+1)B(m-\nu,\lambda+1)}\int_0^\infty p^{s-1}B_p^{\alpha,\beta;\kappa,\mu}(m-\nu,\lambda+1)dp. \end{aligned}$$

Applying Theorem 5.1 to the last integral yields the desired result. □

Theorem 5.3. *Let $m - 1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$, $\Re(s) > 0$ and $|z| < 1$. Then we have*

$$\mathfrak{M}\left[D_z^{\nu,p;\kappa,\mu}\left\{(1-z)^{-\alpha}\right\} : s\right] = \frac{\Gamma^{(\alpha,\beta)}(s)\ z^{-\nu}}{\Gamma(1-\nu)}\sum_{n=0}^\infty \frac{(\alpha)_n}{(1-\nu)_n}\frac{B(m-\nu+s,n+s+1)}{B(m-\nu,n+1)} z^n.$$

Proof. Using the binomial series for $(1 - z)^{-\alpha}$ and Theorem 5.4 with $\lambda = n$ yields

$$\begin{aligned} \mathfrak{M}\left[D_z^{\nu,p;\kappa,\mu}\left\{(1-z)^{-\alpha}\right\} : s\right] &= \mathfrak{M}\left[D_z^{\nu,p;\kappa,\mu}\left\{\sum_{n=0}^\infty \frac{(\alpha)_n}{n!} z^n\right\} : s\right] \\ &= \sum_{n=0}^\infty \frac{(\alpha)_n}{n!}\mathfrak{M}\left[D_z^{\nu,p;\kappa,\mu}\left\{z^n\right\} : s\right] \\ &= \sum_{n=0}^\infty \frac{(\alpha)_n}{n!}\Gamma^{(\alpha,\beta)}(s)\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}\frac{B(m-\nu+s,n+s+1)}{B(m-\nu,n+1)} z^{n-\nu}. \end{aligned}$$

Then the last expression is easily seen to be equal to the desired one. □

Now we present the extended Riemann-Liouville fractional derivative of z^{λ} in terms of the Fox H-function. Let m, n, p, q be integers such that $0 \leq m \leq q$, $0 \leq n \leq p$, and for parameters a_i, $b_i \in \mathbb{C}$ and for parameters α_i, $\beta_j \in \mathbb{R}^+$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$, the H-function is defined in terms of a Mellin-Barnes integral in the following manner ([8, pp. 1–2]; see also [10, p. 343, Definition E.1.] and [15, p. 2, Definition 1.1.]):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}}\Theta(s)\,z^{-s}ds, \tag{5.5}$$

where

$$\Theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \tag{5.6}$$

with the contour $\mathfrak{L}$ suitably chosen, and an empty product, if it occurs, is taken to be unity.

Theorem 5.4. *Let $m-1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$, $\Re(\nu) < \Re(\lambda)$ and $\Re(z) > 0$. Then we have*

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\lambda}\right\} = \frac{\Gamma(\lambda+1)\Gamma(\beta)}{\Gamma(\lambda-\nu+1)B(m-\nu,1+\lambda)\Gamma(\alpha)} \times H_{3,1}^{2,4}\left[p\left|\begin{matrix}(1-\alpha,1),(\lambda+m-\nu+1,\kappa+\mu)\\(0,1),(m-\nu,\mu),(\lambda+1,\kappa),(1-\beta,1)\end{matrix}\right.\right]z^{\lambda-\nu}.$$

Proof. The result can be obtained by taking the inverse Mellin transform of the result in Theorem 3.3 with the aid of (5.5) and (5.6). □

Applying the result in Theorem 3.3 to the Maclaurin series of e^z and the series expressions of the Gauss hypergeometric function ${}_2F_1$ and the Fox-Wright function ${}_p\Psi_q$ gives the extended Riemann-Liouville fractional derivatives of e^z, ${}_2F_1$ and ${}_p\Psi_q(z)$ asserted by the following theorems.

Theorem 5.5. *Let $m-1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$. Then we have*

$$D_z^{\nu,p;\kappa,\mu}\{e^z\} = \frac{z^{-\nu}}{\Gamma(1-\nu)}\sum_{n=0}^{\infty}\frac{1}{(1-\nu)_n}\frac{B_p^{\alpha,\beta;\kappa,\mu}(m-\nu,n+1)}{B(m-\nu,n+1)}z^n \quad (z \in \mathbb{C}).$$

Theorem 5.6. *Let $m-1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$. Then we have*

$$D_z^{\nu,p;\kappa,\mu}\{{}_2F_1(a,b;c;z)\} = \frac{z^{-\nu}}{\Gamma(1-\nu)} \times \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1-\nu)_n}\frac{B_p^{\alpha,\beta;\kappa,\mu}(m-\nu,n+1)\ z^n}{B(m-\nu,n+1)} \quad (|z|<1).$$

Theorem 5.7. *Let $m-1 \leq \Re(\nu) < m$ for some $m \in \mathbb{N}$. Then we have*

$$D_z^{\nu,p;\kappa,\mu}\left\{{}_p\Psi_q\left[\begin{matrix}(a_j,\gamma_j)_{1,p}\\(b_j,\delta_j)_{1,q}\end{matrix};z\right]\right\} = \frac{z^{-\nu}}{\Gamma(1-\nu)}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\gamma_j k)}{\prod_{j=1}^{q}\Gamma(b_j+\delta_j k)} \times \frac{B_p^{\alpha,\beta;\kappa,\mu}(k+1,m-\nu)}{B(k+1,m-\nu)}z^k \quad (|z|<1), \tag{5.7}$$

where ${}_p\Psi_q(z)$ is the Fox-Wright function defined by (see [9, pp. 56–58])

$${}_p\Psi_q(z) = {}_p\Psi_q\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] := \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!}. \tag{5.8}$$

6. ANOTHER APPROACH

In this section we briefly consider another variant of the derivation of the results obtained in the preceding sections. This approach is based on the Cauchy integral formula for the extended fractional derivative operator. We define the extended fractional derivative with respect to an arbitrary, regular and univalent function and calculate the extended fractional derivative of the function $\log z$. Then we determine a representation of the extended fractional derivative operator in terms of the classical fractional derivative operator.

Definition 6.1. Osler [18] was the first to define the derivative of arbitrary order ν by means of the Cauchy integral formula in the form:

$$D_z^\nu \, z^\lambda f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_0^{(z+)} (t-z)^{-\nu-1} t^\lambda f(t)\, dt, \tag{6.1}$$

where the contour shown in Figure 1 consists of a single loop that begins at $t=0$, encloses the point $t=z$ once in the positive direction and returns to $t=0$ without traversing the branch line (the dotted line) for $(t-z)^{-\nu-1}t^\lambda$. This representation is valid for $\nu \in \mathbb{C} \setminus \mathbb{Z}^-$ and $\Re(\lambda) > -1$.

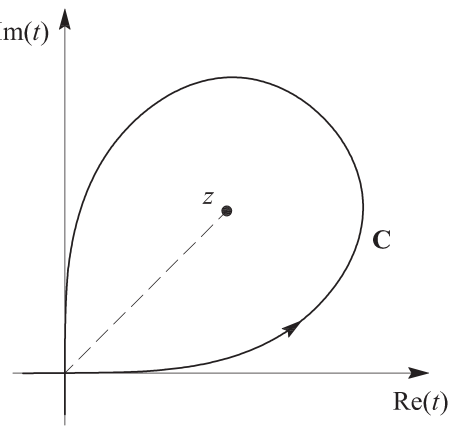

Figure 1: **Branch line** for $t^\lambda (t-z)^{-\nu-1}$

The above representation of the fractional derivative has been very important in the study of fractional calculus and has led to some very interesting new results. Several authors have recently used this approach in their studies (see [2, 3, 4, 17, 19]).

In the sequel, we employ this definition to find the following (presumably) new definition for the extended fractional derivative operator:

Definition 6.2. The extended Riemann-Liouville fractional derivative is defined as

$$D_z^{\nu,p;\kappa,\mu} z^\lambda \, f(z) := \frac{\Gamma(\nu+1)}{2\pi i} \int_0^{(z+)} (z-t)^{-\nu-1} \, t^\lambda \, f(t) {}_1F_1 \left(\alpha; \beta; -\frac{p z^{\kappa+\mu}}{t^\kappa \, (z-t)^\mu} \right) dt, \tag{6.2}$$

where $\Re(\lambda) > -1$, $\Re(p) > 0$, $\Re(\kappa) > 0$ and $\Re(\mu) > 0$.

The special case of (6.2) when $p = 0$ reduces to the fractional derivative operator (6.1). We present an interesting formula for the extended fractional derivative of the function $\log z$ asserted by Theorem 19. For this purpose, we begin by recalling following theorem given by Luo *et al.* [12, Theorem 2.13].

Theorem 6.3. *The extended beta function defined by* (1.1) *possesses the following series expression*

$$\mathcal{B}_p^{(\alpha,\beta;\kappa,\mu)}(x,y) = \sum_{n=0}^{\infty} S_n(1)\, {}_2F_2\left[\begin{array}{c} n+1,\alpha \\ 1,\beta \end{array}; -p\right], \tag{6.3}$$

where $S_n(1)$ *is a polynomial defined by*

$$S_n(x,y;z) := \sum_{j=0}^{n} \frac{(-n)_j}{j!} \frac{\Gamma(x+(j+1)\kappa)\,\Gamma(y+(j+1)\mu)}{\Gamma(x+y+(j+1)(\kappa+\mu))} z^j. \tag{6.4}$$

Theorem 6.4. *Let* $m-1 \leq \Re(\nu) < m$ *for some* $m \in \mathbb{N}$ *and* $\Re(\nu) < \Re(\lambda)$. *Then we have*

$$\begin{aligned} D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda \log z\right\} &= \frac{(\lambda-\nu+1)_m}{\Gamma(m-\nu)} z^{\lambda-\nu} \\ &\times \left[B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)\left(\sum_{k=1}^{m}\frac{1}{\lambda-\nu+k} + \log z\right)\right. \\ &\quad \left. + \sum_{n=0}^{\infty} T_n(\lambda+1,m-\nu;1)\, {}_2F_2\left[\begin{array}{c} n+1,\alpha \\ 1,\beta \end{array}; -p\right]\right], \end{aligned} \tag{6.5}$$

where $\log z$ *is taken it principal branch and* $T_n(\lambda+1,m-\nu;1)$ *is given by*

$$\begin{aligned} T_n(\lambda+1,m-\nu;1) &= \sum_{j=0}^{n} \frac{(-n)_j}{j!} B\left(\lambda+1+(j+1)\kappa,\, m-\nu+(j+1)\mu\right) \\ &\times \left\{\psi\left(\lambda+1+(j+1)\kappa\right) - \psi\left(\lambda+m-\nu+1+(j+1)(\kappa+\mu)\right)\right\} \end{aligned}$$

and $\psi(z) := \Gamma'(z)/\Gamma(z)$ *is the psi (or digamma) function (see, e.g., [23, Section 1.3]).*

Proof. Taking the partial derivative of both sides of (3.3) with respect to λ gives

$$\frac{\partial}{\partial\lambda}\left[D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\}\right] = \frac{\partial f(\lambda)}{\partial\lambda}, \tag{6.6}$$

where

$$f(\lambda) := \frac{\Gamma(\lambda+1)B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)}{\Gamma(\lambda-\nu+1)B(\lambda+1,m-\nu)} z^{\lambda-\nu}.$$

Exchanging the order of the derivative fractional operator and the partial derivative with respect to λ is easily seen to yield

$$\frac{\partial}{\partial\lambda}\left[D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda\right\}\right] = D_z^{\nu,p;\kappa,\mu}\left\{z^\lambda \log z\right\}. \tag{6.7}$$

On the other hand, use (1.3) to express $f(\lambda)$ as follows:

$$f(\lambda) = \frac{\Gamma(\lambda+m-\nu+1)B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)}{\Gamma(\lambda-\nu+1)\,\Gamma(m-\nu)} z^{\lambda-\nu}.$$

Then we differentiate $f(\lambda)$ with respect to λ as follows:

$$\begin{aligned} \Gamma(m-\nu)\frac{\partial f(\lambda)}{\partial\lambda} &= \left\{\frac{\partial}{\partial\lambda}\frac{\Gamma(\lambda+m-\nu+1)}{\Gamma(\lambda-\nu+1)}\right\} B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)\, z^{\lambda-\nu} \\ &+ \frac{\Gamma(\lambda+m-\nu+1)}{\Gamma(\lambda-\nu+1)}\left\{\frac{\partial}{\partial\lambda}B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)\right\} z^{\lambda-\nu} \\ &+ \frac{\Gamma(\lambda+m-\nu+1)}{\Gamma(\lambda-\nu+1)} B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)\left\{\frac{\partial}{\partial\lambda}z^{\lambda-\nu}\right\}. \end{aligned} \tag{6.8}$$

Taking the logarithmic derivative and using a useful identity for the psi function (see, *e.g.*, [23, p. 25, Eq.(7)]) gives

$$\begin{aligned}\frac{\partial}{\partial\lambda}\frac{\Gamma(\lambda+m-\nu+1)}{\Gamma(\lambda-\nu+1)} &= (\lambda-\nu+1)_m\ \{\psi(\lambda+m-\nu+1)-\psi(\lambda-\nu+1)\}\\ &= (\lambda-\nu+1)_m\sum_{k=1}^{m}\frac{1}{\lambda-\nu+k}.\end{aligned}\tag{6.9}$$

Use of the expression (6.3) is seen to yield

$$\begin{aligned}&\frac{\partial}{\partial\lambda}B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)\\ &\quad=\sum_{n=0}^{\infty}T_n(\lambda+1,m-\nu;1)\,{}_2F_2\left[\begin{array}{c}n+1,\alpha\\ 1,\beta\end{array};-p\right].\end{aligned}\tag{6.10}$$

It is easy to see

$$\frac{\partial}{\partial\lambda}z^{\lambda-\nu}=z^{\lambda-\nu}\log z.\tag{6.11}$$

Finally, incorporating the formulas (6.9), (6.10), and (6.11) into (6.8) and considering (6.7) and (6.6) proves the desired identity.

□

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