The asymptotics of a generalised Beta function

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Abstract

We consider the generalised Beta function introduced by Chaudhry et al. [J. Comp. Appl. Math. 78 (1997) 19–32] defined by

\[ B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left[ \frac{-p}{4t(1-t)} \right] dt, \]

where \( \Re(p) > 0 \) and the parameters \( x \) and \( y \) are arbitrary complex numbers. The asymptotic behaviour of \( B(x, y; p) \) is obtained when (i) \( p \) large, with \( x \) and \( y \) fixed, (ii) \( x \) and \( p \) large, (iii) \( x, y \) and \( p \) large and (iv) either \( x \) or \( y \) large, with \( p \) finite. Numerical results are given to illustrate the accuracy of the formulas obtained.

Mathematics Subject Classification: 30E15, 33B15, 34E05, 41A60

Keywords: Generalised Beta function, asymptotic expansion, Mellin-Barnes integral, method of steepest descents

1. Introduction

In [1], Chaudhry et al. introduced a generalised beta function defined by the Euler-type integral\(^1\)

\[ B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left[ \frac{-p}{4t(1-t)} \right] dt, \] (1.1)

where \( \Re(p) > 0 \) and the parameters \( x \) and \( y \) are arbitrary complex numbers. When \( p = 0 \), it is clear that when \( \Re(x) > 0 \) and \( \Re(y) > 0 \) the generalised function reduces to the well-known beta function \( B(x, y) \) of classical analysis. The justification for defining this extension of the beta function is given in [1] and an application of its use in defining extensions of the Gauss and confluent hypergeometric functions is discussed in [2]. It is evident from the definition in (1.1) that \( B(x, y; p) \) satisfies the symmetry property

\[ B(x, y; p) = B(y, x; p). \] (1.2)

A list of useful properties of \( B(x, y; p) \) is detailed by Miller in [4], where it is established that \( B(x, y; p) \) may be expanded as an infinite series of Whittaker functions or Laguerre polynomials; see (A.1). He also obtained a Mellin-Barnes integral representation for

\(^{1}\)The factor 4 is introduced in the exponential for presentational convenience.
B(x, y; p), which we exploit in Section 2, and expressed B(x, x ± n; p) and B(1 ± n, 1; p),
where n is an integer, as finite sums of Whittaker functions.

Our aim in this note is to derive asymptotic expansions for B(x, y; p) for large x, y
and p. We consider (i) |p| → ∞ in |arg p| < 1/4π, with x and y fixed, (ii) x and p large,
(iii) x, y and p large and (iv) either x or y large, with p finite. The expansion for large
p is obtained using a Mellin-Barnes integral representation for B(x, y; p), whereas the
other cases are obtained using the method of steepest descents.

2. The expansion of B(x, y; p) for large p with x, y finite

We start with the Mellin-Barnes integral representation given by Miller [4]

\[ B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma(\frac{x+y+s}{2})\Gamma(x+y+s)} p^{-s} ds \]  

(21)

valid in |arg p| < 1/2π, where c > max{0, −ℜ(x), −ℜ(y)} so that the integration path
lies to the right of all the poles of the integrand situated at s = −k, s = −x − k and
s = −y − k, k = 0, 1, 2, . . . . Displacement of the integration path to the left over
the poles followed by evaluation of the residues (assuming that no two members of the set
{0, x, y} differ by an integer – thereby avoiding the presence of higher-order poles) yields
the result that B(x, y; p) can be expressed as the sum of three 2 F 2 (−1/4p) hypergeometric
functions; see [4, Eq. (1.6)].

Since there are no poles in the half-plane ℜ(s) > c it follows that displacement of the
integration path to the right can produce no algebraic-type asymptotic expansion; see
[8, §5.4]. We can therefore displace the path as far to the right as we please; on such a
displaced path, which we denote by L, the variable |s| is everywhere large. The ratio of
gamma functions in the integrand in (2.1) may then be expanded as an inverse factorial
expansion given by [8, p. 39, Lemma 2.2]

\[ \frac{\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma(\frac{x+y+s}{2})\Gamma(x+y+s)} = \sum_{j=0}^{M-1} (-)^j c_j \Gamma(s-j-\frac{1}{2}) + \rho_M(s)\Gamma(s-M-\frac{1}{2}), \]

where M is a positive integer and ρ_M(s) = O(1) as |s| → ∞ in |arg s| < π. The
coefficients c_j ≡ c_j(x, y) are discussed below where the leading coefficient c_0 = 1.

Substitution of the above inverse factorial expansion into the integral (2.1) then pro-duce

\[ B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} \left\{ \sum_{j=0}^{M-1} (-)^j c_j \frac{1}{2\pi i} \int_L \Gamma(s-j-\frac{1}{2}) p^{-s} ds + R_M \right\}, \]

where

\[ R_M = \frac{1}{2\pi i} \int_L \rho_M(s)\Gamma(s-M-\frac{1}{2}) p^{-s} ds. \]

The integral may be evaluated by the well-known Cauchy-Mellin integral given by (see,
for example, [8, p. 90])

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s+\alpha)z^{-s} ds = z^\alpha e^{-z} \quad (|\arg z| < \frac{1}{2}\pi, \ c > -\Re(\alpha)) \]
to yield

\[ B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} \left\{ p^{-\frac{1}{2}} e^{-p} \sum_{j=0}^{M-1} (-)^j c_j p^{-j} + R_M \right\}. \]

A bound for the remainder \( R_M \) has been considered in [8, p. 71, Lemma 2.7], from which it follows that \( R_M = O(p^{-M-\frac{1}{2}} e^{-p}) \) as \( |p| \to \infty \) in \( |\arg p| < \frac{1}{2} \pi \).

Hence we obtain the asymptotic expansion

\[ B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} p^{-\frac{1}{2}} e^{-p} \left\{ \sum_{j=0}^{M-1} (-)^j c_j p^{-j} + O(p^{-M}) \right\} \tag{2.2} \]

valid as \( |p| \to \infty \) in the sector \( |\arg p| < \frac{1}{2} \pi \). The expansion of \( B(x, y; p) \) for large \( p \) is seen to be exponentially small in \( |\arg p| < \frac{1}{2} \pi \); this is a standard result when there are no poles on the right of the path in (2.1) and routine path displacement does not produce any useful asymptotic information [8, §5.4].

The coefficients \( c_j \) for \( j \geq 1 \) can be generated by the algorithm described in [8, §2.2.4]. It is found that

\[ c_1 = \frac{1}{4}(1 + x + y + 2xy - x^2 - y^2), \]
\[ c_2 = \frac{1}{32}(9 + 6(2 + xy)(x + y + xy) - (7 + 4xy)(x^2 + y^2) - 6(x^3 + y^3) + x^4 + y^4 + 14xy), \]

which are symmetrical in \( x \) and \( y \) as required by (1.2). A closed-form representation for \( c_j \) is derived in the appendix, where it is shown that \( c_j \) can be expressed in terms of a terminating \( 3F_2(1) \) hypergeometric function given by

\[ c_j = c_j(x, y) = \frac{(\frac{1}{2})_j(y + \frac{1}{2})_j}{j!} 3F_2 \left[ -j, \frac{1}{2}y - \frac{1}{2}x, \frac{1}{2}y - \frac{1}{2}x + \frac{1}{2}; 1 \right], \tag{2.3} \]

where \((a)_j = \Gamma(a + j)/\Gamma(a)\) is the Pochhammer symbol. When \( x = y \), this reduces to the simpler expression

\[ c_j(x, x) = \frac{(\frac{1}{2})_j(x + \frac{1}{2})_j}{j!}. \tag{2.4} \]

We remark that the asymptotic expansion of \( B(x, y; p) \) for \( p \to \infty \) could also have been obtained by application of the method of steepest descents, which we shall employ in the subsequent sections. See also the appendix for a different approach.

3. The expansion of \( B(x, y; p) \) for large \( x \) and \( p \) with \( y \) finite

We consider the expansion of \( B(x, y; p) \) for large \( x \) and \( p \), with \( y \) finite, when it is supposed that \( p = ax \), where \( a > 0 \) and \( |\arg x| < \frac{1}{2} \pi \). By the symmetry property (1.2), the same result will also cover the case of large \( y \) and \( p \), with \( x \) finite. From (1.1), we have

\[ B(x, y; ax) = \int_0^1 f(t) e^{-x \psi(t)} dt \quad (|\arg x| < \frac{1}{2} \pi), \tag{3.1} \]

where

\[ \psi(t) = \frac{a}{4t(1-t)} - \log t, \quad f(t) = \frac{(1-t)^y}{t}. \]

Saddle points of the exponential factor are given by \( \psi'(t) = 0 \); that is, at the roots of the cubic

\[ t(1-t)^2 + \frac{1}{4}a(1-2t) = 0. \tag{3.2} \]
We label the three saddles $t_0$, $t_1$ and $t_2$. All three saddles lie on the real axis with $t_0$ situated in the closed interval $[0, 1]$, with $t_1 > 1$ and $t_2 < 0$. The $t$-plane is cut along $(-\infty, 0]$. Paths of steepest descent through the saddles $t_r \ (r = 0, 1)$ are given by

$$\Im\{e^{i\theta}(\psi(t) - \psi(t_r))\} = 0, \quad \theta = \arg x;$$

these paths terminate at $t = 0$ and $t = 1$ in the directions $|\theta - \phi| < \frac{1}{2} \pi$ and $\frac{1}{2} \pi < \theta - \phi < \frac{3}{2} \pi$, respectively, where $\phi = \arg t$.

When $x > 0$, the integration path coincides with the steepest descent path over the saddle $t_0$; for complex $x$ in the sector $|\arg x| < \frac{1}{2} \pi$, the steepest descent path through $t_0$ becomes deformed but still terminates at $t = 0$ and $t = 1$; see Fig. 1. Application of the saddle-point method then yields the leading behaviour

$$B(x, y; ax) \sim \sqrt{\frac{2\pi}{x\psi''(t_0)}} f(t_0) e^{-x\psi(t_0)}$$

$$= \sqrt{\frac{2\pi}{x\psi''(t_0)}} t_0^{x-1}(1 - t_0)^{y-1} \exp\left[\frac{-ax}{4t_0(1 - t_0)}\right] \quad (3.3)$$

as $|x| \to \infty$ in the sector $|\arg x| < \frac{1}{2} \pi$, where some routine algebra combined with (3.2) shows that

$$\psi''(t_0) = \frac{1 - 3t_0 + 4t_0^2}{t_0^2(1 - t_0)(2t_0 - 1)}.$$ 

We remark that the saddle $t_0 \equiv t_0(a)$ has to be computed for a particular value of the parameter $a$, either directly from (3.2) or as a cubic root.

The asymptotic expansion of $B(x, y; ax)$ is given by [7, p. 47]

$$B(x, y; ax) \sim 2e^{-x\psi(t_0)} \sum_{n=0}^{\infty} \frac{C_{2n}\Gamma(n + \frac{1}{2})}{x^{n + \frac{1}{2}}} \quad (|x| \to \infty, \ |\arg x| < \frac{1}{2} \pi). \quad (3.4)$$

Figure 1: The steepest descent and ascent paths through the saddles $t_0$ and $t_1$ (heavy dots) when $a = 1/3$ and (a) $\theta = 0$ and (b) $\theta = \pi/4$. The arrows indicate the integration path. In (b) the steepest ascent paths spiral round $t = 0$ out to infinity passing onto adjacent Riemann surfaces. The saddle $t_2$ on the branch cut on $(-\infty, 0]$ is not shown.
The coefficients $C_n$ can be obtained by an inversion process and are listed for $n \leq 8$ in [3, p. 119] and for $n \leq 4$ in [9, p. 13]. Alternatively, they can be obtained by an expansion process to yield Wojdylo’s formula [10] given by

$$C_n = \frac{1}{2a_0^{(n+1)/2}} \sum_{k=0}^{n} b_{n-k} \sum_{j=0}^{k} \frac{(-1)^j (\frac{1}{2} n + \frac{1}{2})_j}{j! a_0^j} B_{kj}; \quad (3.5)$$

see also [5, 6]. Here $B_{kj} \equiv B_{kj}(a_1, a_2, \ldots, a_{k-j+1})$ are the partial ordinary Bell polynomials generated by the recursion\(^2\)

$$B_{kj} = \sum_{r=1}^{k-j+1} a_r B_{k-r,j-1}, \quad B_{k0} = \delta_{k0},$$

where $\delta_{mn}$ is the Kronecker symbol, and the coefficients $a_r$ and $b_r$ appear in the expansions

$$\psi(t) - \psi(t_0) = \sum_{r=0}^{\infty} a_r (t - t_0)^{r+2}, \quad f(t) = \sum_{r=0}^{\infty} b_r (t - t_0)^r \quad (3.6)$$

valid in a neighbourhood of the saddle $t = t_0$.

In numerical computations we choose a value of the parameter $a$ and compute the saddle $t_0$ from (3.2). With a value of $y$, Mathematica is used to determine the coefficients $a_r$ and $b_r$ for $0 \leq r \leq n_0$. The coefficients $C_{2n}$ can then be calculated for $0 \leq n \leq n_0$ from (3.5). We display the computed values of $C_{2n}$ for different values of $a$ and $y$ in Table 1. In Table 2, the values of the absolute relative error in the computation of $B(x, y; ax)$ from (3.4) are presented as a function of the truncation index $n$ when $x = 100$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a = 1, \ y = 1$</th>
<th>$a = \frac{1}{2}, \ y = \frac{3}{2}$</th>
<th>$a = \frac{3}{2}, \ y = \frac{5}{4}$</th>
<th>$a = 2, \ y = \frac{1}{2}$</th>
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<tbody>
<tr>
<td>0</td>
<td>$+0.2668661228$</td>
<td>$+0.1364219142$</td>
<td>$+0.2036093538$</td>
<td>$+0.3909054941$</td>
</tr>
<tr>
<td>1</td>
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<td>$+0.2683838462$</td>
<td>$+0.0762869817$</td>
<td>$-0.0309094064$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.0635656655$</td>
<td>$-0.1085963949$</td>
<td>$-0.0456489054$</td>
<td>$-0.0039290992$</td>
</tr>
<tr>
<td>3</td>
<td>$+0.0186002666$</td>
<td>$+0.0151339630$</td>
<td>$+0.0137423943$</td>
<td>$+0.002409801$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.0039253710$</td>
<td>$-0.0003383888$</td>
<td>$-0.0026770977$</td>
<td>$-0.000515807$</td>
</tr>
<tr>
<td>5</td>
<td>$+0.0012059654$</td>
<td>$+0.0004533741$</td>
<td>$+0.0003423270$</td>
<td>$+0.0000299402$</td>
</tr>
</tbody>
</table>

### 4. The expansion of $B(x, y; p)$ for large $x$, $y$ and $p$

We consider the expansion of $B(x, y; p)$ for large $x$, $y$ and $p$, when it is supposed that $p = ax$ and $y = bx$, where $a > 0$, $b > 0$ and $|\arg x| < \frac{1}{2} \pi$. From (1.1), we have

$$B(x, y; p) = \int_0^1 f(t) e^{-x\psi(t)} dt \quad (|\arg x| < \frac{1}{2} \pi), \quad (4.1)$$

\(^2\)For example, this generates the values $B_{41} = a_4, B_{42} = a_2^2 + 2a_1a_3, B_{43} = 3a_2^2a_2$ and $B_{44} = a_1^4$. 
Table 2: Values of the absolute relative error in $B(x, y; ax)$ when $x = 100$ for different truncation index.

<table>
<thead>
<tr>
<th>n</th>
<th>$a = 1$, $y = 1$</th>
<th>$a = \frac{1}{2}$, $y = \frac{3}{2}$</th>
<th>$a = \frac{3}{2}$, $y = \frac{5}{2}$</th>
<th>$a = 2$, $y = \frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.838 \times 10^{-3}$</td>
<td>$9.682 \times 10^{-3}$</td>
<td>$1.853 \times 10^{-3}$</td>
<td>$3.963 \times 10^{-4}$</td>
</tr>
<tr>
<td>1</td>
<td>$1.770 \times 10^{-5}$</td>
<td>$5.892 \times 10^{-5}$</td>
<td>$1.666 \times 10^{-5}$</td>
<td>$7.426 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.295 \times 10^{-7}$</td>
<td>$2.058 \times 10^{-7}$</td>
<td>$1.255 \times 10^{-7}$</td>
<td>$1.153 \times 10^{-8}$</td>
</tr>
<tr>
<td>3</td>
<td>$9.506 \times 10^{-10}$</td>
<td>$1.517 \times 10^{-10}$</td>
<td>$8.562 \times 10^{-10}$</td>
<td>$8.568 \times 10^{-11}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.295 \times 10^{-11}$</td>
<td>$9.526 \times 10^{-12}$</td>
<td>$5.011 \times 10^{-12}$</td>
<td>$2.332 \times 10^{-13}$</td>
</tr>
<tr>
<td>5</td>
<td>$3.688 \times 10^{-12}$</td>
<td>$1.933 \times 10^{-13}$</td>
<td>$5.472 \times 10^{-14}$</td>
<td>$6.917 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

where

$$
\psi(t) = \frac{a}{4t(1-t)} - \log t - b \log(1-t), \quad f(t) = \frac{1}{t(1-t)}.
$$

(4.2)

Saddle points of the exponential factor are given by the roots of the cubic

$$
t(1-t)\{1-(b+1)t\} + \frac{1}{4}a(1-2t) = 0.
$$

(4.3)

Routine examination of this cubic shows that, when $a > 0$, $b > 0$, all roots are real, with one root greater than 1, one in the interval $[0, 1]$ and one negative root. The distribution of the saddles is thus similar to that in Section 3, where we continue to label the saddle situated in $[0, 1]$ by $t_0$. The topology of the path of steepest descent through the saddle $t_0$, given by $\Im\{e^{\theta t}(\psi(t) - \psi(t_0))\} = 0$ where $\theta = \arg x$, is also similar to that depicted in Fig. 1.

Accordingly, the expansion of $B(x, y; p)$ when $p = ax$ and $y = bx$, with $a > 0$, $b > 0$, is given by

$$
B(x, bx; ax) \sim 2e^{-x\psi(t_0)} \sum_{n=0}^{\infty} C_{2n} \Gamma\left(n + \frac{1}{2}\right) \frac{\Gamma(n + 1)}{x^{n+\frac{1}{2}}} \quad (|x| \to \infty, \ 0 < \arg x < \frac{1}{2}\pi),
$$

(4.4)

where the coefficients $C_{2n}$ can be determined from (3.5) when the coefficients $a_r$ and $b_r$ in (3.6) are evaluated from the definitions of $\psi(t)$ and $f(t)$ in (4.2).

The leading behaviour is

$$
B(x, bx; ax) \sim \sqrt{\frac{2\pi}{x\psi''(t_0)}} f(t_0) e^{-x\psi(t_0)}
$$

$$
= \sqrt{\frac{2\pi}{x\psi''(t_0)}} t_0^{-x-1} (1 - t_0)^{bx-1} \exp\left[\frac{-ax}{4t_0(1-t_0)}\right]
$$

(4.5)

as $|x| \to \infty$ in the sector $|\arg x| < \frac{1}{2}\pi$, where

$$
\psi''(t_0) = \frac{1 - 3t_0 + 4t_0^2}{t_0^2(1-t_0)(2t_0-1)} \left(1 - \frac{bt_0}{1-t_0}\right) + \frac{b}{t_0(1-t_0)^2}
$$

and $t_0 \equiv t_0(a, b)$ is the root of (4.3) situated in $t \in [0, 1]$.

We note that when $b = 1$ we have the result [1, 4]

$$
B(x, x; p) = 2^{1-2x} \pi^{\frac{1}{2}} p^{(x-1)/2} e^{-\frac{1}{2}p} W_{-\frac{1}{2}, \frac{1}{2}x} (p)
$$

in terms of the Whittaker function $W_{\kappa, \mu}(z)$; see (A.1).
5. The behaviour of $B(x, y; p)$ for large $x$ and finite $y$ and $p$

In this final section, we examine the behaviour of $B(x, y; p)$ for large complex $x = |x|e^{i\theta}$, with $0 \leq \theta \leq \pi$, when $y$ and $p > 0$ are finite. The situation when $-\pi \leq \theta \leq 0$ is analogous and, in the case of real $y$, $B(x, y; p)$ assumes conjugate values. This case has been discussed in [2, Appendix], but is repeated (with minor corrections) here for completeness. By the symmetry property (1.2), the same result will also cover the case of large $y$, with $x$ and $p$ finite.

From (1.1), we have upon interchanging $x$ and $y$ (by virtue of (1.2))

$$B(x, y; p) = \int_0^1 f(t)e^{-|x|\psi(t)}dt,$$

where

$$\psi(t) = \frac{\alpha}{t(1-t)} - e^{i\theta} \log(1-t), \quad f(t) = \frac{t^{y-1}}{1-t}, \quad \alpha := \frac{p}{4|x|}.$$  

Because $p > 0$ is a fixed parameter, the integral (5.1) is valid for arbitrary complex values of $x$ and $y$. Saddle points of the exponential factor arise when $\psi'(t) = 0$; that is, when

$$t^2(1-t) + \alpha e^{-i\theta}(1-2t) = 0.$$

We label the three saddles $t_0$, $t_1$ and $t_2$ as in Section 3. When $\theta = 0$, all three saddles are situated on the real axis with $t_0 \in [0, 1]$ and $t_1 > 1$, $t_2 < 1$. As $\theta$ increases, the saddles $t_0$ and $t_2$ rotate about the origin and $t_1$ rotates about the point $t = 1$. The result of this rotation is that, when $\theta = \pi$, $t_0$ and $t_2$ become a complex conjugate pair near the origin and $t_1$ is situated in the interval $[0, 1]$; see Fig. 2.

![Figure 2: The steepest descent and ascent paths through the saddles $t_0$ and $t_1$ (heavy dots) when $\alpha = 1/3$ and (a) $\theta = 0.25\pi$, (b) $\theta = \theta_0 = 0.65595\pi$, (c) $\theta = 0.69\pi$, (d) $\theta = \theta_1 = 0.71782\pi$, (e) $\theta = 0.80\pi$ and (f) $\theta = \pi$. The arrows indicate the integration path. The steepest ascent paths spiral round $t = 1$ out to infinity passing onto adjacent Riemann surfaces. The saddle $t_2$ is not shown. The $t$-plane is cut along $[1, \infty)$.](image-url)
When \( \theta = 0 \), the integration path coincides with the steepest descent path passing over the saddle \( t_0 \) given approximately by
\[
t_0 \simeq \alpha^{\frac{1}{2}} - \frac{1}{2} \alpha \quad (x \to \infty).
\]
Then, with the estimates
\[
x \psi(t_0) \simeq (px)^{1/2} + \frac{3}{8p}, \quad \psi''(t_0) \simeq 2\alpha^{-\frac{1}{2}},
\]
we find by application of the saddle-point method the leading behaviour
\[
B(x, y; p) \sim \sqrt{\frac{\pi}{x}} \left( \frac{p}{4x} \right)^{\frac{1}{2}y - \frac{1}{4}} \exp \left[ -\left( px \right)^{1/2} - \frac{3}{8p} \right] \quad (\theta = 0, \ x \to +\infty).
\] (5.4)

When \( \theta = \pi \), we find from (5.3) that the saddle \( t_1 \) close to the point \( t = 1 \) is given by
\[
t_1 \simeq 1 - \alpha + \alpha^3 \quad (|x| \to \infty)
\]
and
\[
|x| \psi(t_1) \simeq |x| + \frac{1}{4}\alpha - |x| \log \alpha, \quad \psi''(t_1) \simeq \alpha^{-2}.
\]
The integration path again coincides with the steepest descent path through \( t_1 \), and so we obtain the behaviour
\[
B(x, y; p) \sim i \sqrt{\frac{2\pi}{|x|}} \left( \frac{p}{4|y|} \right)^{\frac{1}{2}|y|} \exp \left[ -\frac{1}{4p} |x| \right] \quad (\theta = \pi, \ x \to -\infty)
\]
\[
= \sqrt{\frac{2\pi}{|x|}} \left( \frac{p}{4|x|} \right)^{-|x|} \exp \left[ |x| - \frac{1}{4\alpha} \right].
\] (5.5)

The detailed terms in (5.4) and (5.5) were given in [2, Appendix].

A detailed study of the topology of the steepest descent paths\(^3\) through the saddles \( t_0 \) and \( t_1 \) when \( 0 \leq \theta \leq \pi \) is summarised in Fig. 2 for the particular case \( \alpha = \frac{1}{3} \). The \( t \)-plane is cut along \([1, \infty)\) and paths of steepest descent either terminate at \( t = 0 \) (with \( |arg t| < \frac{1}{2} \pi \)), \( t = 1 \) (with \( |arg(1 - t)| < \frac{1}{2} \pi \)) or at infinity. Paths that approach infinity spiral round the point \( t = 1 \) passing onto adjacent Riemann surfaces. The figures reveal that there are two critical values of the phase \( \theta \), where the saddles \( t_0 \) and \( t_1 \) become connected (via a Stokes phenomenon). We denote these values by \( \theta_0 \equiv \theta_0(\alpha) \) and \( \theta_1 \equiv \theta_1(\alpha) \), where \( \alpha \) is defined in (5.2). The values of these critical angles are tabulated in Table 3 for different \( \alpha \).

When \( 0 \leq \theta < \theta_0(\alpha) \), the integration path can be deformed to coincide with the steepest descent path passing over \( t_0 \), so that the leading behaviour in (5.4) applies in this sector. When \( \theta_0(\alpha) < \theta < \theta_1(\alpha) \), the integration path is deformed to pass over both saddles \( t_0 \) and \( t_1 \), where each steepest descent path spirals out to infinity. Finally, when \( \theta_1(\alpha) < \theta \leq \pi \), the integration path is deformed to pass over only the saddle \( t_1 \).

Based on these considerations and on the approximation of the saddles \( t_0 \simeq \alpha^{\frac{1}{2}} - \frac{1}{2} \alpha' \), \( t_1 \simeq 1 + \alpha' - \alpha^3 \), where \( \alpha' = p/(4x) \), the leading behaviour of \( B(x, y; p) \) is found to be
\[
B(x, y; p) \sim \begin{cases}
J_0 & 0 \leq \theta < \theta_1(\alpha) \\
J_0 - J_1 & \theta_1(\alpha) < \theta < \theta_2(\alpha) \\
J_1 & \theta_2(\alpha) < \theta \leq \pi
\end{cases}
\] (5.6)

\(^3\)The saddle \( t_2 \) does not enter into our consideration as it plays no role in the asymptotic evaluation of \( B(x, y; p) \) when \( 0 \leq \theta \leq \pi \).
as $|x| \to \infty$ when $0 \leq \theta \leq \pi$ (with $y$ and $p > 0$ finite), where

$$J_0 := \sqrt{\frac{2\pi}{|x| \psi''(t_0)}} t_0^{y-1} (1 - t_0)^{x-1} \exp \left[ -\frac{p}{4t_0(1 - t_0)} \right] \quad (5.7)$$

and

$$J_1 := \sqrt{\frac{2\pi}{|x| \psi''(t_1)}} t_1^{y-1} (1 - t_1)^{x-1} \exp \left[ -\frac{p}{4t_1(1 - t_1)} \right] \quad (5.8)$$

with $\arg \psi''(t_r) \in [0, 2\pi], r = 0, 1$. Inspection of Table 3 shows that as $\alpha$ decreases (that is, as $|x|$ increases for fixed $p$) the angular sector $\theta_0(\alpha) \leq \theta \leq \theta_1(\alpha)$, where $B(x, y; p)$ receives a contribution from both saddles, increases. We also show in Table 3 the value of $\theta = \theta^*(\alpha)$ at which $\Re(\psi(t_0)) = \Re(\psi(t_1))$ when the saddles are of the same height. We have $\theta_0(\alpha) < \theta^*(\alpha) < \theta_1(\alpha)$; then, for $\theta < \theta^*(\alpha)$ the saddle $t_0$ is dominant, whereas when $\theta > \theta^*(\alpha)$ the saddle $t_1$ is dominant in the large-$|x|$ limit.

In Table 4 we present the results of numerical calculations using the asymptotic behaviour of $B(x, y; p)$ in (5.6) compared to the values obtained by numerical integration of (5.1). The parameter values chosen correspond to $\alpha = 0.01$ and the saddles $t_0$ and $t_1$ are computed from (5.3), with the leading forms $J_0$ and $J_1$ computed from (5.7) and (5.8). It is seen from Table 3 that the exchange of dominance between the two contributory saddles arises for $\theta \simeq 0.60\pi$.

**Appendix: A closed-form expression for the coefficients $c_j$**

In this appendix we derive a closed-form expression for the coefficients $c_j$ appearing in the expansion (2.2). Miller [4, Eq. (2.3a)] has shown that $B(x, y; p)$ can be expressed as a convergent series of Whittaker functions in the form

$$B(x, y; p) = 2^{1-x-y} \pi \frac{1}{2} p^{(y-1)/2} e^{-\frac{1}{2} p} \sum_{k=0}^{\infty} \frac{(\frac{1}{2} y - \frac{1}{2} x) k (\frac{1}{2} + \frac{1}{2} y - \frac{1}{2} x) k}{k!} W_{-k-\frac{1}{2} y, \frac{1}{2} y}(p), \quad (A.1)$$
Table 4: Values of the asymptotic behaviour of $B(x, y; p)$ in (5.6) with the calculated value when $|x| = 50$, $p = 2$ ($\alpha = 0.01$) and $y = \frac{1}{2}$ for different $\theta = \arg x$.

<table>
<thead>
<tr>
<th>$\theta / \pi$</th>
<th>Asymptotic value</th>
<th>Calculated value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$+5.175 \times 10^{-06}$</td>
<td>$+5.187 \times 10^{-06}$</td>
</tr>
<tr>
<td>0.20</td>
<td>$-8.210 \times 10^{-06} + 2.081 \times 10^{-06}i$</td>
<td>$-8.223 \times 10^{-06} + 2.096 \times 10^{-06}i$</td>
</tr>
<tr>
<td>0.40</td>
<td>$+3.468 \times 10^{-05} - 6.934 \times 10^{-06}i$</td>
<td>$+3.470 \times 10^{-05} - 7.020 \times 10^{-06}i$</td>
</tr>
<tr>
<td>0.50</td>
<td>$+2.647 \times 10^{-06} - 9.853 \times 10^{-05}i$</td>
<td>$+2.402 \times 10^{-06} - 9.855 \times 10^{-05}i$</td>
</tr>
<tr>
<td>0.60</td>
<td>$-8.837 \times 10^{-04} - 3.821 \times 10^{-03}i$</td>
<td>$-8.781 \times 10^{-04} - 3.823 \times 10^{-03}i$</td>
</tr>
<tr>
<td>0.70</td>
<td>$-5.944 \times 10^{+28} + 1.659 \times 10^{+28}i$</td>
<td>$-5.952 \times 10^{+28} + 1.652 \times 10^{+28}i$</td>
</tr>
<tr>
<td>0.80</td>
<td>$+2.786 \times 10^{+54} + 3.451 \times 10^{+54}i$</td>
<td>$+2.786 \times 10^{+54} + 3.459 \times 10^{+54}i$</td>
</tr>
<tr>
<td>1.00</td>
<td>$+4.146 \times 10^{+77}$</td>
<td>$+4.154 \times 10^{+77}$</td>
</tr>
</tbody>
</table>

where $W_{k, \mu}(x)$ is the Whittaker function. For $p \to \infty$ with bounded $k$, we have the expansion [7, Eq. (13.19.3)]

$$W_{-k-\frac{1}{2}, \frac{1}{2}}y(p) = p^{-k-\frac{1}{2}}e^{-\frac{1}{2}p}\left\{ \sum_{n=0}^{N-1} \frac{(-)^n (\frac{1}{2} + k)_n(y + \frac{1}{2} + k)_n}{n! p^n} + O(p^{-N}) \right\},$$

where $N$ is a positive integer. Then we obtain from (A.1)

$$B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} p^{-\frac{1}{2}} e^{-p}\left\{ S(x, y; p) + O(p^{-N}) \right\},$$

(A.2)

where

$$S(x, y; p) = \sum_{k=0}^{N-1} \frac{(\frac{1}{2}y - \frac{1}{2}x)_k(\frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x)_k}{k! p^k} \sum_{n=0}^{N-1} \frac{(-)^n (\frac{1}{2} + k)_n(y + \frac{1}{2} + k)_n}{n! p^n}$$

$$= \sum_{k=0}^{N-1} \frac{(\frac{1}{2}y - \frac{1}{2}x)_k(\frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x)_k}{k!} \sum_{j=k}^{N-1} \frac{(-)^{j-k} (\frac{1}{2} + k)_{j-k}(y + \frac{1}{2} + k)_{j-k}}{(j-k)! p^j} + O(p^{-N})$$

and we have made the change of summation index $n \to j - k$. Use of the fact that $(-j)_k = (-)^k j! / (j - k)!$, the above double sum can be written as

$$\sum_{k=0}^{N-1} \frac{(\frac{1}{2}y - \frac{1}{2}x)_k(\frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x)_k}{k!} \sum_{j=k}^{N-1} \frac{(-)^j j! \Gamma(j + \frac{1}{2}) \Gamma(y + j + \frac{1}{2})}{j! \Gamma(k + \frac{1}{2}) \Gamma(y + \frac{1}{2} + k)}$$

$$= \sum_{j=0}^{N-1} \frac{(-)^j j!}{j! p^j} \sum_{k=0}^{j} \frac{(-)^j k(\frac{1}{2}y - \frac{1}{2}x)_k(\frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x)_k}{k! (\frac{1}{2})_k (y + \frac{1}{2})_k}$$

$$= \sum_{j=0}^{N-1} \frac{(-)^j j!}{j! p^j} 3F_2\left[ -j, \frac{1}{2}y - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x; \frac{1}{2}, y + \frac{1}{2} \right].$$

(A.3)

upon reversal of the order of summation and identification of the inner sum over $k$ as a terminating $3F_2$ series of unit argument.

Comparison of (A.2) and (A.3) with the expansion obtained in (2.2) then yields the final result

$$c_j = \frac{(\frac{1}{2})_j y + (\frac{1}{2})_j}{j!} 3F_2\left[ -j, \frac{1}{2}y - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x; \frac{1}{2}, y + \frac{1}{2} \right].$$

(A.4)
References


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