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SERIES REPRESENTATIONS OF THE REMAINDERS IN THE EXPANSIONS FOR CERTAIN FUNCTIONS WITH APPLICATIONS

CHAO-PING CHEN* AND RICHARD B. PARIS

ABSTRACT. We present a summary of the series representations of the remainders in the expansions in ascending powers of t of $2/(e^t+1)$, $\operatorname{sech} t$ and $\coth t$ and establish simple bounds for these remainders when $t>0$. Several applications of these expansions are given which enable us to deduce some inequalities and completely monotonic functions associated with the ratio of two gamma functions. In addition, we derive a (presumably new) quadratic recurrence relation for the Bernoulli numbers B_n.

1. Introduction

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined, respectively, by the generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad (|t|<2\pi)\quad\text{and}\quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad (|t|<\pi).$$

The numbers $B_n=B_n(0)$ and $E_n=2^nE_n(\frac{1}{2})$, which are known to be rational numbers and integers, respectively, are called Bernoulli and Euler numbers.

It follows from [25, Chapter 4, Part I, Problem 154] that

$$\sum_{j=1}^{2m}\frac{B_{2j}}{(2j)!}t^{2j}<\frac{t}{e^t-1}-1+\frac{t}{2}<\sum_{j=1}^{2m+1}\frac{B_{2j}}{(2j)!}t^{2j} \tag{1.1}$$

for $t>0$ and $m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathbb{N}:=\{1,2,3,\ldots\}$. The inequality (1.1) can be also found in [12, 26]. It is also known [33, p. 64] that for integer $m\geq 2$,

$$\frac{t}{e^t-1}-1+\frac{t}{2}=\sum_{j=1}^{m-1}\frac{B_{2j}}{(2j)!}\,t^{2j}+(-1)^{m-1}t^{2m}s_m(t), \tag{1.2}$$

where

$$s_m(t)=\sum_{k=1}^{\infty}\frac{2}{(2k\pi)^{2m-2}(t^2+(2k\pi)^2)}.$$

It is easily seen that (1.2) implies (1.1).

Binet's first formula [32, p. 16] for the logarithm of $\Gamma(x)$ states that

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}+\int_0^{\infty}\left(\frac{t}{e^t-1}-1+\frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t\quad (x>0). \tag{1.3}$$

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Combining (1.2) with (1.3), Xu and Han [38] deduced in 2009 that for every $m \in \mathbb{N}_0$, the function

$$R_m(x) = (-1)^m \left[\ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \ln \sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right] \tag{1.4}$$

is completely monotonic on $(0, \infty)$. Recall that a function $f(x)$ is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I, \quad n \in \mathbb{N}_0). \tag{1.5}$$

For $m = 0$, the complete monotonicity property of $R_m(x)$ was proved by Muldoon [21]. Alzer [2] first proved in 1997 that $R_m(x)$ is completely monotonic on $(0, \infty)$. In 2006, Koumandos [12] proved the double inequality (1.1), and then used (1.1) and (1.3) to give a simpler proof of the complete monotonicity property of $R_m(x)$. In 2009, Koumandos and Pedersen [13, Theorem 2.1] strengthened this result.

Chen and Paris [9, Lemma 1] presented an analogous result to (1.1) given by

$$\sum_{j=2}^{2m+1} \frac{(1-2^{2j})B_{2j}}{j} \frac{t^{2j-1}}{(2j-1)!} < \frac{2}{e^t+1} - 1 + \frac{t}{2} < \sum_{j=2}^{2m} \frac{(1-2^{2j})B_{2j}}{j} \frac{t^{2j-1}}{(2j-1)!} \tag{1.6}$$

for $t > 0$ and $m \in \mathbb{N}$. This inequality can also be written for $t > 0$ and $m \in \mathbb{N}_0$ as

$$(-1)^{m+1} \left(\frac{2}{e^t+1} - 1 - \sum_{j=1}^{m} \frac{(1-2^{2j})B_{2j}}{j} \frac{t^{2j-1}}{(2j-1)!} \right) > 0. \tag{1.7}$$

Based on the inequality (1.7), Chen and Paris [9, Theorem 1] proved that for every $m \in \mathbb{N}_0$, the function

$$F_m(x) = (-1)^m \left[\ln \left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right) - \frac{1}{2} \ln x - \sum_{j=1}^{m} \left(1 - \frac{1}{2^{2j}}\right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right] \tag{1.8}$$

is completely monotonic on $(0, \infty)$. This result is similar to the complete monotonicity property of $R_m(x)$ in (1.4). In analogy with (1.2), these authors also considered [9, Eq. (2.4)] the remainder in the truncated expansion in ascending powers of t of the function $2/(e^t+1)$ and gave an integral representation for this remainder when $t > 0$. Similar expansions for $2/(e^t+1)$ and $\operatorname{sech} t$ have recently been obtained by Koumandos in [15, Section 4].

In this paper, we summarize the series representations of the remainders in the expansions of $2/(e^t+1)$, $\coth t$ and $\operatorname{sech} t$ and establish simple bounds for these remainders when $t > 0$. We also obtain the double inequality for $t > 0$ and $m \in \mathbb{N}_0$,

$$\sum_{j=0}^{2m+1} \frac{E_{2j}}{(2j)!} t^{2j} < \operatorname{sech} t < \sum_{j=0}^{2m} \frac{E_{2j}}{(2j)!} t^{2j}. \tag{1.9}$$

Several applications of these results are presented in Section 3, where some complete monotonicity results and a double inequality for a ratio of factorials are considered. Finally, in Section 4, we derive a quadratic recurrence relation for the Bernoulli numbers given in Theorem 4.1.

2. **Main results**

In this section we summarize the expansions of three hyperbolic functions given by the following theorem:

Theorem 2.1. *For $t > 0$ and integer $m \geq 1$, we have the expansions*

$$\frac{2}{e^t+1} = 1 + \sum_{j=1}^{m-1} \frac{2(1-2^{2j})B_{2j}}{(2j)!}\, t^{2j-1} + (-1)^m t^{2m-1} p_m(t), \tag{2.1}$$

$$\coth t = \sum_{j=0}^{m-1} \frac{2^{2j}B_{2j}}{(2j)!}\, t^{2j-1} + (-1)^{m-1} t^{2m-1} q_m(t), \tag{2.2}$$

$$\frac{1}{\cosh t} = \sum_{j=0}^{m-1} \frac{E_{2j}}{(2j)!}\, t^{2j} + (-1)^m t^{2m} r_m(t), \tag{2.3}$$

where the remainder functions $p_m(t)$, $q_m(t)$ and $r_m(t)$ are defined by

$$p_m(t) = \sum_{k=0}^{\infty} \frac{4}{[(2k+1)\pi]^{2m-2}(t^2+[(2k+1)\pi]^2)}, \tag{2.4}$$

$$q_m(t) = \sum_{k=1}^{\infty} \frac{2}{(k\pi)^{2m-2}(t^2+(k\pi)^2)}, \qquad r_m(t) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{[(k+\frac{1}{2})\pi]^{2m-1}(t^2+[(k+\frac{1}{2})\pi]^2)}. \tag{2.5}$$

Proofs of the expansions in (2.1) and (2.3) are given in [15, Section 4]. The expansion (2.2) follows in a straightforward manner from (1.2) upon noting that

$$\coth t = \frac{e^t+e^{-t}}{e^t-e^{-t}} = 1 + \frac{2}{e^{2t}-1}$$

and making the change of variable $t \to 2t$.

Corollary 2.1. *For $t > 0$ and integer $m \geq 1$, the expansions in Theorem 2.1 can be expressed in the form*

$$\frac{2}{e^t+1} = 1 + \sum_{j=1}^{m-1} \frac{2(1-2^{2j})B_{2j}}{(2j)!}\, t^{2j-1} + \frac{2(1-2^{2m})B_{2m}}{(2m)!}\, t^{2m-1}\Theta_1(t,m), \tag{2.6}$$

$$\coth t = \sum_{j=0}^{m-1} \frac{2^{2j}B_{2j}}{(2j)!}\, t^{2j-1} + \frac{2^{2m}B_{2m}}{(2m)!}\, t^{2m-1}\Theta_2(t,m), \tag{2.7}$$

$$\frac{1}{\cosh t} = \sum_{j=0}^{m-1} \frac{E_{2j}}{(2j)!}\, t^{2j} + \frac{E_{2m}}{(2m)!}\, t^{2m}\Theta_3(t,m), \tag{2.8}$$

where $0 < \Theta_r(t,m) < 1$ for $r = 1, 2, 3$.

Proof. To show (2.6) we express the remainder $p_m(t)$ in (2.4) as

$$p_m(t) = \frac{4\Theta_1(t,m)}{\pi^{2m}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m}}, \qquad \Theta_1(t,m) = \frac{F_1(t)}{F_1(0)},$$

where

$$F_1(t) := \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m-2}(t^2+[(2k+1)\pi]^2)}.$$

Since

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2m}}=(1-2^{-2m})\zeta(2m)=(1-2^{-2m})\frac{(2\pi)^{2m}}{2(2m)!}|B_{2m}|,$$

we find that

$$p_m(t)=\frac{2(2^{2m}-1)}{(2m)!}\,|B_{2m}|\Theta_1(t,m)=(-1)^m\,\frac{2(1-2^{2m})B_{2m}}{(2m)!}\,\Theta_1(t,m)$$

upon use of the fact that $B_{2m}=(-1)^{m-1}|B_{2m}|$. It is clear that $F_1(t)>0$ and is a decreasing function on $[0,\infty)$. Hence, for all $t>0$ and integer $m\geq 1$, we have $0<F_1(t)<F_1(0)$ and thus $0<\Theta_1(t,m)<1$, thereby establishing (2.6).

For (2.7), we write the remainder $q_m(t)$ in (2.5) as

$$q_m(t)=\frac{2\Theta_2(t,m)}{\pi^{2m}}\sum_{k=1}^{\infty}\frac{1}{k^{2m}},\qquad \Theta_2(t,m)=\frac{F_2(t)}{F_2(0)},$$

where

$$F_2(t):=\sum_{k=1}^{\infty}\frac{1}{k^{2m-2}(t^2+(k\pi)^2)}.$$

It is well known that

$$\sum_{k=1}^{\infty}\frac{1}{k^{2m}}=\frac{(-1)^{m-1}(2\pi)^{2m}B_{2m}}{2(2m)!}$$

and hence

$$q_m(t)=\frac{2^{2m}B_{2m}}{(2m)!}\,t^{2m-1}\Theta_2(t,m).$$

By the same reasoning as above it follows that $0<F_2(t)<F_2(0)$ and thus, for all $t>0$ and integer $m\geq 1$, we have $0<\Theta_2(t,m)<1$, thereby establishing (2.7).

Finally, the remainder $r_m(t)$ in (2.5) can be written as

$$r_m(t)=\frac{2\Theta_3(t,m)}{(\frac{1}{2}\pi)^{2m+1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2m+1}},\qquad \Theta_3(t,m)=\frac{F_3(t)}{F_3(0)},$$

where where

$$F_3(t):=\sum_{k=0}^{\infty}(-1)^kA_k,\qquad A_k:=\frac{1}{(2k+1)^{2m-1}(t^2+[(2k+1)\pi/2]^2)}.$$

From [1, p. 807], we have

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2m+1}}=\frac{(-1)^m\pi^{2m+1}}{2^{2m+2}(2m)!}\,E_{2m}$$

and hence

$$r_m(t)=\frac{(-1)^mE_{2m}}{(2m)!}\,\Theta_3(t,m).$$

Then it is easily seen that $A_{2k}>A_{2k+1}$ for $k\in N_0$ and integer $m\geq 1$; thus $F_3(t)>0$ for $t>0$. Differentiation yields

$$F_3'(t)=-2t\sum_{k=0}^{\infty}\frac{(-1)^kA_k}{t^2+[(2k+1)\pi/2]^2}$$

and a similar reasoning shows that $F_3'(t) < 0$ for $t > 0$. Hence, for all $t > 0$ and integer $m \geq 1$, we have $0 < F_3(t) < F_3(0)$ and thus $0 < \Theta_3(t, m) < 1$. The proof of Corollary 2.1 is complete.□

Remark 2.1. *From* (2.1) *we retrieve* (1.7).

Corollary 2.2. *For $t > 0$ and $m \in \mathbb{N}$, we have*

$$(-1)^m \left(\frac{2e^t}{(e^t+1)^2} - \sum_{j=1}^{m} \frac{(2^{2j}-1)B_{2j}}{j \cdot (2j-2)!} t^{2j-2} \right) > 0. \tag{2.9}$$

Proof. Differentiating the expression in (2.1) with m replaced by $m+1$, we find

$$-\frac{2e^t}{(e^t+1)^2} = -\sum_{j=1}^{m} \frac{(2^{2j}-1)B_{2j}}{j \cdot (2j-2)!} t^{2j-2} + (-1)^{m+1} \big(t^{2m+1} p_{m+1}(t)\big)'.$$

It is easy to see that

$$t^2 p_{m+1}(t) + p_m(t) = \frac{4}{\pi^{2m}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m}} = \frac{4}{\pi^{2m}} (1 - 2^{-2m}) \zeta(2m),$$

where $\zeta(z)$ is the Riemann zeta function. This last expression can be written as

$$t^2 p_{m+1}(t) = \frac{4}{\pi^{2m}} (1 - 2^{-2m}) \zeta(2m) - p_m(t). \tag{2.10}$$

Then, since $p_m(t)$ is strictly decreasing for $t > 0$, we deduce from (2.10) that $t^2 p_{m+1}(t)$ is strictly increasing for $t > 0$. Hence, $t^{2m+1} p_{m+1}(t)$ is strictly increasing for $t > 0$, and we then obtain

$$(-1)^m \left(\frac{2e^t}{(e^t+1)^2} - \sum_{j=1}^{m} \frac{(2^{2j}-1)B_{2j}}{j \cdot (2j-2)!} t^{2j-2} \right) = \big(t^{2m+1} p_{m+1}(t)\big)' > 0$$

for $t > 0$ and $m \in \mathbb{N}$. The proof is complete. □

Remark 2.2. *From [22, p. 592, Eq. (24.7.9)] and [34, p. 43, Ex. 12(i)] we have*

$$E_{2n}(x) = (-1)^n \sin(\pi x) \int_0^\infty \frac{4t^{2n} \cosh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} \, dt \qquad (0 < x < 1, \quad n \in \mathbb{N}_0),$$

from which it follows that

$$E_{4m}(x) > 0 \quad \text{and} \quad E_{4m+2}(x) < 0 \qquad (0 < x < 1, \quad m \in \mathbb{N}_0).$$

Use of this result enables us to deduce (1.9) *from* (2.8)*. Note that the inequality* (1.9) *can also be written as*

$$(-1)^{m+1} \left(\operatorname{sech} t - \sum_{j=0}^{m} \frac{E_{2j}}{(2j)!} t^{2j} \right) > 0 \qquad (t > 0, \; m \in \mathbb{N}_0). \tag{2.11}$$

3. **Miscellaneous results**

In this section we present several results that can be deduced from the expansions given in Section 2.

3.1. **The proof of a conjecture of Chen.**

In [6], Chen proposed the following conjecture.

Conjecture 3.1.. *For $t > 0$ and $m \in \mathbb{N}_0$, let*

$$\nu_m(t) = \frac{e^{t/4} - e^{3t/4}}{e^t - 1} - \sum_{j=0}^{m} \frac{2B_{2j+1}(\frac{1}{4})}{(2j+1)!} t^{2j},$$

where $B_n(x)$ denotes the Bernoulli polynomials. Then, for $t > 0$ and $m \in \mathbb{N}_0$,

$$(-1)^m \nu_m(t) > 0. \tag{3.1}$$

Chen [6, Lemma 1] has proved the statement in (3.1) for $m = 0, 1, 2,$ and 3. He has also pointed out in [6] that, if (3.1) is true, then it follows that the function

$$V_m(x) = (-1)^m \left\{ \ln \frac{\Gamma(x + \frac{3}{4})}{x^{1/2}\Gamma(x + \frac{1}{4})} - \sum_{j=1}^{m} \frac{B_{2j+1}(\frac{1}{4})}{j(2j+1)} \frac{1}{x^{2j}} \right\} \tag{3.2}$$

for $m \in \mathbb{N}_0$ is completely monotonic on $(0, \infty)$.

It was shown in [6] that $\nu_m(t)$ can be written as

$$\nu_m(t) = -\frac{1}{2\cosh(\frac{t}{4})} + \sum_{j=0}^{m} \frac{E_{2j}}{2(2j)!} \left(\frac{t}{4}\right)^{2j}$$

and (3.1) is equivalent to (2.11). Hence, for $t > 0$ and $m \in \mathbb{N}_0$, it follows that $(-1)^m \nu_m(t) > 0$ holds true.

It was also shown in [6] that

$$\begin{aligned} V_m(x) &= (-1)^m \left\{ \int_0^\infty \left(\frac{e^{t/4} - e^{3t/4}}{e^t - 1} + \frac{1}{2} \right) \frac{e^{-xt}}{t} \,\mathrm{d}t - \sum_{j=1}^{m} \frac{2B_{2j+1}(\frac{1}{4})}{(2j+1)!} \int_0^\infty t^{2j-1} e^{-xt} \mathrm{d}t \right\} \\ &= \int_0^\infty (-1)^m \nu_m(t) \frac{e^{-xt}}{t} \,\mathrm{d}t. \end{aligned} \tag{3.3}$$

We then obtain from (3.3) that for all $m \in \mathbb{N}_0$,

$$(-1)^n V_m^{(n)}(x) = \int_0^\infty (-1)^m \nu_m(t) t^{n-1} e^{-xt} \,\mathrm{d}t > 0$$

for $x > 0$ and $n \in \mathbb{N}_0$. Hence, the function $V_m(x)$, defined by (3.2), is completely monotonic on $(0, \infty)$.

3.2. **A double inequality for a ratio of factorials.**

Noting [6, Eq. (3.26)] that $B_{2n+1}(\frac{1}{4})$ can be expressed in terms of the Euler numbers

$$B_{2n+1}(\tfrac{1}{4}) = -\frac{(2n+1)E_{2n}}{4^{2n+1}} \qquad (n \in \mathbb{N}_0),$$

we find that (3.2) can be written as

$$V_m(x) = (-1)^m \left\{ \ln \frac{\Gamma(x + \frac{3}{4})}{x^{1/2}\Gamma(x + \frac{1}{4})} + \sum_{j=1}^{m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}} \right\}.$$

From the inequality $V_m(x) > 0$ for $x > 0$, we then obtain the following

Corollary 3.1. *For $x > 0$, we have*

$$x^{1/2} \exp\left(-\sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}}\right) < \frac{\Gamma(x+\frac{3}{4})}{\Gamma(x+\frac{1}{4})} < x^{1/2} \exp\left(-\sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}}\right). \tag{3.4}$$

The problem of finding new and sharp inequalities for the gamma function $\Gamma(x)$ and, in particular, for the Wallis ratio[1]

$$\frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}$$

has attracted the attention of many researchers (see [8, 9, 14, 16, 17, 19] and references therein). For example, Chen and Qi [8] proved that for $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \tag{3.5}$$

where the constants $(4/\pi) - 1$ and $\frac{1}{4}$ are the best possible. This inequality is a consequence of the complete monotonicity on $(0, \infty)$ of the function (see [7])

$$\Upsilon(x) = \frac{\Gamma(x+1)}{\sqrt{x+\frac{1}{4}}\,\Gamma(x+\frac{1}{2})}.$$

If we write (3.4) as

$$\frac{1}{\sqrt{x}} \exp\left(\sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}}\right) < \frac{\Gamma(x+\frac{1}{4})}{\Gamma(x+\frac{3}{4})} < \frac{1}{\sqrt{x}} \exp\left(\sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}}\right)$$

and replace x by $x + \frac{1}{4}$, we find

$$\frac{1}{\sqrt{x+\frac{1}{4}}} \exp\left(\sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x+\frac{1}{4})^{2j}}\right) < \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}$$
$$< \frac{1}{\sqrt{x+\frac{1}{4}}} \exp\left(\sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x+\frac{1}{4})^{2j}}\right). \tag{3.6}$$

Noting that (3.5) holds, we then deduce from (3.6) that

$$\frac{1}{\sqrt{\pi(x+\frac{1}{4})}} \exp\left(\sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x+\frac{1}{4})^{2j}}\right) < \frac{(2n-1)!!}{(2n)!!}$$
$$< \frac{1}{\sqrt{\pi(x+\frac{1}{4})}} \exp\left(\sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x+\frac{1}{4})^{2j}}\right), \tag{3.7}$$

which generalizes a recent result of Chen [6, Eq. (3.40)], who proved the inequality (3.7) for $m = 1$.

We remark that Mortici et al. [20] proved that some functions associated with the products $\prod_{k=1}^{n} \frac{3k-2}{3k}$ and $\prod_{k=1}^{n} \frac{3k-1}{3k}$ are completely monotonic and established some sharp inequalities.

[1]Here, we employ the special double factorial notation $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!$, $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1) = 2^n \pi^{-1/2} \Gamma(n+\frac{1}{2})$; see [1, p. 258].

3.3. Integral representations for $\ln\pi$.

Sondow and Hadjicostas [31] introduced and studied the generalized-Euler-constant function $\gamma(z)$, defined by

$$\gamma(z)=\sum_{n=1}^{\infty}z^{n-1}\left(\frac{1}{n}-\ln\frac{n+1}{n}\right),$$

where the series converges when $|z|\leq 1$. Pilehrood and Pilehrood [24] considered the function $z\gamma(z)$ ($|z|\leq 1$). The function $\gamma(z)$ generalizes both Euler's constant $\gamma(1)$ and the alternating Euler constant $\gamma(-1)=\ln(4/\pi)$ [29,30]. An interesting comparison by Sondow [29] is the double integral and alternating series

$$\ln\frac{4}{\pi}=\int_0^1\int_0^1\frac{x-1}{(1+xy)\ln(xy)}\mathrm{d}x\mathrm{d}y=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\ln\frac{n+1}{n}\right).$$

The formula (3.3) can provide integral representations for the constant π. For example, the choice $(x,m)=(1/4,0)$ in (3.3) yields

$$\int_0^\infty\left(\frac{e^{t/4}-e^{3t/4}}{e^t-1}+\frac{1}{2}\right)\frac{2e^{-t/4}}{t}\mathrm{d}t=\ln\frac{4}{\pi}, \tag{3.8}$$

which provides a new integral representation for the alternating Euler constant $\gamma(-1)=\ln(4/\pi)$. The choice $(x,m)=(3/4,0)$ in (3.3) yields

$$\int_0^\infty\left(\frac{e^{t/4}-e^{3t/4}}{e^t-1}+\frac{1}{2}\right)\frac{2e^{-3t/4}}{t}\mathrm{d}t=\ln\frac{\pi}{3}.$$

Many formulas exist for the representation of π, and a collection of these formulas is listed in [27,28]. For more history of π see [3,4,10].

3.4. The complete monotonicity of a remainder function in the Barnes G-function. The following expansion for Barnes G-function was established by Ferreira and López [11, Theorem 1]

$$\ln G(z+1)=\frac{1}{4}z^2+z\ln\Gamma(z+1)-\left(\frac{1}{2}z^2+\frac{1}{2}z+\frac{1}{12}\right)\ln z-\ln A$$
$$+\sum_{k=1}^{N-1}\frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}}+\mathcal{R}_N(z)\qquad(N\in\mathbb{N})$$

for $|z|\to\infty$ in $|\arg z|<\pi$, where B_{2k+2} are the Bernoulli numbers and A is the Glaisher–Kinkelin constant with the numerical value $1.282427\ldots$. For $\Re(z)>0$, the remainder $\mathcal{R}_N(z)$ is given by

$$\mathcal{R}_N(z)=\int_0^\infty\left(\frac{t}{e^t-1}-\sum_{k=0}^{2N}\frac{B_k}{k!}t^k\right)\frac{e^{-zt}}{t^3}dt. \tag{3.9}$$

Estimates for $|\mathcal{R}_N(z)|$ were also obtained by Ferreira and López [11], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative axis. Pedersen [23, Theorem 1.1] proved that for any $N\geq 1$, the function $x\mapsto(-1)^N\mathcal{R}_N(x)$ is completely monotonic on $(0,\infty)$.

Here, we present another proof of this complete monotonicity result. From (2.2), we obtain the following inequality:

$$\sum_{j=0}^{2m} \frac{2^{2j}B_{2j}}{(2j)!}t^{2j-1} < \coth t < \sum_{j=0}^{2m+1} \frac{2^{2j}B_{2j}}{(2j)!}t^{2j-1} \qquad (t>0,\ m\in\mathbb{N}_0),$$

which is equivalent to

$$(-1)^N \left(\coth t - \sum_{j=0}^{N} \frac{2^{2j}B_{2j}}{(2j)!}t^{2j-1}\right) > 0 \qquad (t>0,\ N\in\mathbb{N}_0).$$

Replacement of t by $t/2$ in the above inequality yields

$$(-1)^N \left(\frac{t}{2}\coth\left(\frac{t}{2}\right) - \sum_{j=0}^{N} \frac{B_{2j}}{(2j)!}t^{2j}\right) > 0 \qquad (t>0,\ N\in\mathbb{N}_0).$$

Accordingly, we obtain from (3.9) that the function

$$(-1)^N \mathcal{R}_N(x) = \int_0^\infty (-1)^N \left(\frac{t}{2}\coth\left(\frac{t}{2}\right) - \sum_{k=0}^{N} \frac{B_{2k}}{(2k)!}t^{2k}\right) \frac{e^{-xt}}{t^3}\,dt \tag{3.10}$$

is completely monotonic on $(0,\infty)$.

4. A quadratic recurrence relation for B_n

Euler (see [22, p. 595, Eq. (24.14.2)] and [37]) presented a quadratic recurrence relation for the Bernoulli numbers:

$$\sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} = (1-n)B_n - nB_{n-1} \qquad (n \geq 1),$$

which is equivalent to[2]

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j}B_{2n-2j} = -(2n+1)B_{2n} \qquad (n\geq 2). \tag{4.1}$$

Other quadratic recurrences for the Bernoulli numbers have been given by Gosper (see [35, Eq. (38)]) as

$$B_n = \frac{1}{1-n}\sum_{k=0}^{n}(1-2^{1-k})(1-2^{k-n+1})\binom{n}{k} B_k B_{n-k}$$

and by Matiyasevitch [18] (see also [37]) as

$$B_n = \frac{1}{n(n+1)}\sum_{k=2}^{n-2}\left\{n+2-2\binom{n+2}{k}\right\} B_k B_{n-k} \qquad (n\geq 4).$$

Here, we present a (presumably new) quadratic recurrence relation for the Bernoulli numbers.

[2] The relation (4.1) can be used to show by induction that $(-1)^{n-1}B_{2n} > 0$ for all $n \geq 1$; that is, the even-index Bernoulli numbers have alternating signs.

Theorem 4.1. *The Bernoulli numbers satisfy the following quadratic recurrence relation:*

$$B_n = \frac{1}{2^n-1}\sum_{k=2}^{n-2}(1-2^k)\binom{n}{k}B_kB_{n-k} \qquad (n\geq 4). \tag{4.2}$$

Proof. If we replace t by $t/2$ in (2.1), we find

$$\frac{2}{e^{t/2}+1} = 1+\sum_{j=2}^{\infty}b_jt^{j-1}, \qquad b_j = \frac{2(1-2^j)B_j}{2^{j-1}\cdot j!}. \tag{4.3}$$

The Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}B_n\frac{t^n}{n!}, \tag{4.4}$$

which yields

$$\frac{t/2}{e^{t/2}-1} = \sum_{k=0}^{\infty}\frac{B_kt^k}{2^kk!}. \tag{4.5}$$

It then follows from (4.3) and (4.5) that

$$\begin{aligned}\frac{t}{e^t-1} &= \left(1+\sum_{j=2}^{\infty}b_jt^{j-1}\right)\sum_{k=0}^{\infty}\frac{B_kt^k}{2^kk!}\\ &= \sum_{k=0}^{\infty}\frac{B_kt^k}{2^kk!}+\sum_{j=2}^{\infty}b_jt^{j-1}\sum_{k=0}^{\infty}\frac{B_kt^k}{2^kk!}\\ &= \sum_{j=0}^{\infty}\frac{B_jt^j}{2^jj!}+\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}b_{k+2}\frac{B_{j-k-1}t^k}{2^{j-k-1}(j-k-1)!},\end{aligned}$$

that is

$$\frac{t}{e^t-1} = \sum_{j=0}^{\infty}\left(\frac{B_j}{2^jj!}+\sum_{k=0}^{j-1}b_{k+2}\frac{B_{j-k-1}}{2^{j-k-1}(j-k-1)!}\right)t^j. \tag{4.6}$$

Equating coefficients of equal powers of t in (4.4) and (4.6), we see that

$$\frac{B_j}{j!} = \frac{B_j}{2^j\cdot j!}+\sum_{k=0}^{j-1}b_{k+2}\frac{B_{j-k-1}}{2^{j-k-1}\cdot(j-k-1)!} \qquad (j\in\mathbb{N}_0). \tag{4.7}$$

Substitution of the coefficients b_j in (4.3) into (4.7) then yields

$$B_j = \frac{j!}{2^j-1}\sum_{k=0}^{j-1}\frac{2(1-2^{k+2})B_{k+2}B_{j-k-1}}{(k+2)!\cdot(j-k-1)!} \qquad (j\in\mathbb{N}).$$

It is easy to see that

$$\begin{aligned}B_j &= \frac{j!}{2^j-1}\left(\sum_{k=0}^{j-3}\frac{2(1-2^{k+2})B_{k+2}B_{j-k-1}}{(k+2)!\cdot(j-k-1)!}-\frac{(1-2^j)B_j}{j!}+\frac{2(1-2^{j+1})B_{j+1}}{(j+1)!}\right)\\ &= \frac{j!}{2^j-1}\sum_{k=0}^{j-3}\frac{2(1-2^{k+2})B_{k+2}B_{j-k-1}}{(k+2)!\cdot(j-k-1)!}+B_j+\frac{2(1-2^{j+1})B_{j+1}}{(2^j-1)(j+1)}.\end{aligned}$$

We therefore obtain

$$B_{j+1} = \frac{1}{2^{j+1}-1}\sum_{k=0}^{j-3}(1-2^{k+2})\binom{j+1}{k+2}B_{k+2}B_{j-k-1} \qquad (n \in \mathbb{N}\setminus\{1,2\}),$$

which, upon replacing j by $n-1$ and k by $k-2$, yields (4.2). This completes the proof of Theorem 4.1. □

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